

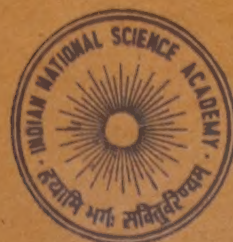
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## SELECTION OF OPTIMAL SITE FOR NEW DEPOT OF SPECIFIED CAPACITY WITH TWO OBJECTIVES

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*(Received 15 October 1987)*

The problem of selecting an optimal site for a new depot of specified capacity from several potential sites along with determining an optimal schedule to take buses from depots to the starting points of their routes and also determining the spare capacity available at each of the depots after the construction of a new depot is considered. Capacities of the respective depots and the number of buses required at the starting points of the respective routes are specified. The problem has two objectives—one primary and another secondary. The primary objective is to minimize the capital expenditure to be incurred in constructing a new depot plus the present value of the expenditure to be incurred in total dead mileage over a planning horizon. The secondary objective is to minimize the maximum of the dead mileage of individual buses. An algorithm is developed to obtain the solution of this two-objective location of problem.

### 1. INTRODUCTION

The problem of selection of an optimal site for a new depot from several potential sites is of current interest. For depots built in the past cannot provide parking facilities for all buses because of increase in their number with the passage of time and depots have reached upper limit of augmentation of their capacities. In the past, this problem has been tackled by applying certain rudimentary techniques based on commonsense and experience. But this has resulted at times into expenses which are avoidable. Owing to this and awareness that even a small percentage of savings in urban transportation schemes can result into substantial savings because of huge investments involved in them, there is need to minimize unnecessary expenditure by all means. One of the ways in which unnecessary expenditure can be minimized is to employ analytical tools to deal with the problems of urban bus transportation. Recently Sharma and Prakash<sup>3</sup> have applied analytical tools for optimizing dead mileage in



urban bus routes. The present paper deals with the problem of selecting an optimal site for a new depot of specified capacity from several potential sites along with determining an optimal schedule to take buses from depots to the starting points of their routes and also determining the spare capacity available at each of the depots after the construction of a new depot. Capacities of the respective depots and the number of buses required at the starting points of the respective routes are specified. The problem has two objectives—one primary and another secondary. The primary objective is to minimize the capital expenditure to be incurred in constructing a new depot plus the present value of the expenditure to be incurred in total dead mileage over a planning horizon. The secondary objective is to minimize the maximum of the dead mileage of individual buses. The term dead mileage refers to the distance traversed by a vehicle when no service is provided. For instance, the distance traversed by a bus from a depot where it is parked over-night to the starting point of its route in the morning is dead mileage because this distance is traversed without severing any useful purpose. An algorithm is developed to obtain the solution of the two-objective location problem and is illustrated through a numerical example.

## 2. FORMULATION OF THE PROBLEM

Suppose that there are  $m$  existing depots,  $s$  potential sites for a new depot and  $n$  starting points of routes. The total number of buses required at the starting points of the various routes is greater than the number of buses to be parked overnight at the depots necessitating the construction of a new depot. Capacity of the new depot is so specified that there is space at depots for parking a certain number of additional buses besides providing parking facilities for all buses operating on routes after its construction. For the convenience of notation, the proposed new depot at potential site  $l$  ( $l = 1, \dots, s$ ) will be designated as depot  $m + l$ . Let  $a_i$  ( $i = 1, \dots, m + s$ ) be the maximum number of buses that can be parked overnight at depot  $i$ ,  $b_j$  ( $j = 1, \dots, n$ ) be the number of buses required at the starting point of route  $j$ ,  $b_{n+1}$  be the number of additional buses apart from buses operating on routes which can be parked overnight at depots after the construction of a new depot,  $c_{m+l}$  ( $l = 1, \dots, s$ ) units be the cost of creating parking facility for one bus at site  $l$ ,  $d_{ij}$  ( $i = 1, \dots, m + s$ ;  $j = 1, \dots, n$ ) units be the distance from depot  $i$  to the starting point of route  $j$ ,  $k_{m+l}$  ( $l = 1, \dots, s$ ) units be the initial setup cost of constructing depot at site  $l$ ,  $N$  be the number of years over which the planning horizon is spread,  $p$  units be the expenditure incurred in traversing one unit of dead mileage by a bus,  $r$  be the rate of interest per annum,  $\lambda_{m+l}$  ( $l = 1, \dots, s$ ) be an integer assuming value 0 or 1 according as site  $l$  is not selected or selected for new depot,  $x_{ij}$  ( $i = 1, \dots, m + s$ ;  $j = 1, \dots, n$ ) be the number of buses to be sent from depot  $i$  to the starting point of route  $j$ , and  $x_{i(n+1)}$  ( $i = 1, \dots, m + s$ ) be the number of buses apart from buses operating on routes which can be parked overnight at depot  $i$  after the construction of a new depot. The capacity of each proposed new depot at potential site is identical and is equal to  $\sum_{j=1}^{n+1} b_j - \sum_{i=1}^m a_i$ . It



is required to select an optimal site for a new depot from the potential sites along with to determine an optimal schedule to take buses from depots to the starting points of their routes and also to determine the spare capacity available at each of the depots after the construction of a new depot subject to the constraints of the problem. The primary objective is to minimize the capital expenditure to be incurred in constructing a new depot plus the present value of the expenditure to be incurred in total dead mileage over the entire planning horizon. The secondary objective is to minimize the maximum distance among the distances traversed by individual buses from depots to the starting points of their respective routes. The mathematical formulation of this two-objective location problem is as follows: Find  $\lambda_{m+l} = 0$  or 1 ( $l = 1, \dots, s$ ) and  $x_{ij} \geq 0$  ( $i = 1, \dots, m + s; j = 1, \dots, n + 1$ ) which minimize

$$C = \left\{ \begin{aligned} & \sum_{l=1}^s \lambda_{m+l} [k_{m+l} + (\sum_{j=1}^{n+1} b_j - \sum_{i=1}^m a_i) c_{m+l} + \sum_{j=1}^n c_{(m+l)j} x_{(m+l)j}] \\ & + \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \end{aligned} \right\} \quad \dots(1)$$

where

$$c_{ij} = 2 (365) p d_{ij} \sum_{t=1}^N (1 + r/100)^{-t} \quad (i = 1, \dots, m + s; j = 1, \dots, n) \quad \dots(2)$$

and

$$D = \max \{d_{ij} : x_{ij} > 0 \quad (i = 1, \dots, m + s; j = 1, \dots, n)\} \quad \dots(3)$$

according to priorities in the order of their occurrence subject to the constraints

$$\sum_{l=1}^s \lambda_{m+l} = 1 \quad \dots(4)$$

$$\sum_{j=1}^{n+1} x_{ij} = a_i \quad (i = 1, \dots, m) \quad \dots(5)$$

$$\sum_{l=1}^s \sum_{j=1}^{n+1} \lambda_{m+l} x_{(m+l)j} = \sum_{j=1}^{n+1} b_j - \sum_{i=1}^m a_i \quad \dots(6)$$

$$\sum_{i=1}^m x_{ij} + \sum_{l=1}^s \lambda_{m+l} x_{(m+l)j} = b_j \quad (j = 1, \dots, n + 1). \quad \dots(7)$$

### 3. SOLUTION PROCEDURE

The two-objective location problem formulated above is a mixed integer non-linear problem. A procedure is outlined to obtain the solution of this problem. As



$\lambda_{m+l}$ 's are integers assuming values 0 or 1 and are subjected to the constraint (4), it follows that all elements except one in each distinct set of values assumed by  $\lambda_{m+1}$ , ...,  $\lambda_{m+s}$  are zero and that the nonzero element is 1; so the total number of distinct sets of values assumed by  $\lambda_{m+1}$ , ...,  $\lambda_{m+s}$  is only  $s$ . Setting  $\lambda_{m+1} = 1$  and  $\lambda_{m+l} = 0$  ( $l = 2, \dots, s$ ), the two-objective location problem given by equations (1) through (7) yields the following problem. Find  $x_{ij} \geq 0$  ( $i = 1, \dots, m, m+1; j = 1, \dots, n+1$ ) which minimize

$$C_1 = k_{m+l} + \left( \sum_{j=1}^{n+1} b_j - \sum_{i=1}^m a_i \right) c_{m+1} \sum_{i=1}^{m, m+1} \sum_{j=1}^n c_{ij} x_{ij} \quad \dots(8)$$

where

$$c_{ij} = 2 (365) p d_{ij} \sum_{t=1}^N (1 + r/100)^{-t} \quad (i = 1, \dots, m, m+1; j = 1, \dots, n) \quad \dots(9)$$

and

$$D_1 = \max \{d_{ij} : x_{ij} > 0 \quad (i = 1, \dots, m, m+1; j = 1, \dots, n)\} \quad \dots(10)$$

according to priorities in the order of their occurrence subject to the constraints (5),

$$\sum_{j=1}^{n+1} x_{(m+l)j} = \sum_{j=1}^{n+1} b_j - \sum_{i=1}^m a_i \quad \dots(11)$$

$$\sum_{i=1}^{m, m+1} x_{ij} = b_j \quad (j = 1, \dots, n+1). \quad \dots(12)$$

The problem described by equations (8), (9), (10), (5), (11), (12) is designated as the 1st two-objective problem of the given two-objective location problem. The problem obtained from the given two-objective location problem by setting  $\lambda_{m+2} = 1$  and  $\lambda_{m+l} = 0$  ( $l = 1, 3, \dots, s$ ) is designated as the two-objective problem of the given two-objective location problem, and so on. Finally, the problem obtained from the given two-objective location problem by setting  $\lambda_{m+s} = 1$  and  $\lambda_{m+l} = 0$  ( $l = 1, \dots, s-1$ ) is designated as the  $s$ th two-objective problem of the given two-objective location problem. These two-objective problems of the given two-objective location problem are of similar type and are reduced to equivalent single objective transportation-type problems following the procedure developed by Prakash<sup>2</sup>. The single-objective transportation-type problem equivalent to the  $l$ th two-objective problem of the given two-objective location problem is designated as the  $l$ -th single objective transportation-type problem of the given two-objective location problem. Among all the single-objective transportation-type problems of the given two-objective location problem, the one whose objective function has minimum value provides complete solution of the given two-objective location problem and the optimal site for a new depot is the site corresponding to it.



The procedure to reduce the 1st two-objective problem of the given two-objective location problem to the 1st single-objective transportation-type problem of the given two-objective location problem is briefly described below. First, the set  $\{d_{ij} : i = 1, \dots, m, m+1; j = 1, \dots, n\}$  is partitioned into subsets  $L_k$  ( $k = 1, \dots, q$ ) in the following way. Each of the subsets  $L_k$  consists of the  $d_{ij}$ 's having the same numerical value,  $L_1$  consists of the  $d_{ij}$ 's having the largest numerical value,  $L_2$  consists of the  $d_{ij}$ 's having the next largest numerical value, and so on. Finally,  $L_q$  consists of the  $d_{ij}$ 's having the smallest numerical value. After this, priority factors,  $M_0, M_1, \dots, M_q$  are assigned to

$$C_1, \sum_{L_1} x_{ij}, \dots, \sum_{L_q} x_{ij}$$

respectively. Here  $\sum_{L_k} x_{ij}$  is the sum of the  $x_{ij}$ 's corresponding to the  $d_{ij}$ 's belonging to  $L_k$ . The priority factors  $M_k$ 's are all positive and are such that the expression  $\sum_{k=0}^q \alpha_k M_k$  has the same sign as the nonzero  $\alpha_k$  with the smallest subscript present in it irrespective of the values of other  $L_k$ 's. Having done all this, the 1st two-objective problem of the given two-objective location problem is reduced to the equivalent 1st single-objective transportation-type problem of the given two-objective location problem seeking to determine  $x_{ij} \geq 0$  ( $i = 1, \dots, m, m+1; j = 1, \dots, n+1$ ) which minimize

$$\begin{aligned} Z_1 = M_0 [k_{m+1} + (\sum_{j=1}^{n+1} b_j - \sum_{i=1}^m a_i) c_{m+1}] + \sum_{i=1}^{m, m+1} \sum_{j=1}^n c_{ij} x_{ij} \\ + \sum_{k=1}^q M_k \sum_{L_k} x_{ij} \end{aligned} \quad \dots(13)$$

subject to the equations (9), (5), (11), (12). This 1st single-objective transportation-type problem is amenable to solution by the standard transportation method discussed by Hadley<sup>1</sup>. The minimum value of the total cost  $C_1$  comprising the capital expenditure to be incurred in constructing a new depot at potential site 1 plus the present value of the expenditure to be incurred in total dead mileage over the planning horizon is obtained by the coefficient of  $M_0$  in the expression on the right-hand side of eqn. (13) after it has been minimized. And if  $M_u$  is the priority factor with the smallest subscript among all the priority factors assigned to the nonzero sums  $\sum_{L_k} x_{ij}$  in the expression on the right-hand side of eqn. (13) after it has been minimized, the minimum of the maximum distance  $D_1$  among the distances traversed by individual buses from the depots to the starting points of their respective routes is given by the value of the  $d_{ij}$ 's belonging to  $L_u$ .

#### 4. NUMERICAL EXAMPLE

Now the above procedure is applied to obtain the optimal solution of a numerical problem which is obtained by taking  $b_{6+1} = 5$ ,  $c_{2+1} = 2000$ ,  $c_{2+2} = 2100$ ,  $k_{2+1}$



$=200000$ ,  $k_{2+2} = 250000$ ,  $m = 2$ ,  $n = 6$ ,  $N = 20$ ,  $p = 6$ ,  $r = 10$ ,  $s = 2$ , and assigning numerical values to all other quantities in the location problem formulated in Section 2. The tableau representation of the numerical problem is shown in Table I.

In this Table, rows with the headings 'Depot 1' and 'Depot 2' refer to existing depots while entries in these rows in the column with the heading of 'Capacities of depots' refer to their capacities. And rows with the headings 'Depot (2 + 1)' and 'Depot (2 + 2)' refer to depots at potential sites while entries in these rows in the column with the heading "Capacities of depots" refer to their capacities which are

TABLE I

*Capacities of depots and buses required at starting points of routes and distances from depots to starting point of routes*

	Starting points of						Capacities of depots
	route1	route2	route3	route4	route5	route6	
Depot 1	2	1	.5	5	4	3	25
Depot 2	1	2	3	3	5	2	20
Depot (2 + 1)	3	.5	1	2	3	4	$53+5-45=11$
Depot (2 + 2)	2.5	2	1	3	2	3	$51+5-45=11$
Buses required	12	10	8	6	7	8	

TABLE II

*Tableau providing optimal solution of 1st single-objective transportation-type problem of numerical problem*

	Starting points of						Spare capacity of depots available
	route 1	route 2	route 3	route 4	route 5	route 6	
Depot 1	$74578.8M_0$ $+M_4$ 10	$37289.4M_0$ $+M_5$ 8	$18644.7M_0$ $+M_6$ 8	$186447.0M_0$ $+M_1$ 2	$149157.6M_0$ $+M_2$ 2	$111868.2M_0$ $+M_3$ 5	0 25 5
Depot 2	$37289.4M_0$ $+M_5$ 12	$74578.8M_0$ $+M_4$ 12	$111868.2M_0$ $+M_3$ 0	$111868.2M_4$ $+M_3$ 0	$186447.0M_0$ $+M_1$ 8	$74578.8M_0$ $+M_4$ 8	0 20 8
Depot (2+1)	$111868.2M_0$ $+M_3$	$18644.7M_0$ $+M_6$	$37289.4M_0$ $+M_5$	$74578.8M_0$ $+M_4$ 6	$111868.2M_0$ $+M_3$ 5	$149157.6M_0$ $+M_2$	0 11
Buses required	12	10	8	6	7	8	5



identical and equal to  $51 + 5 - 45 = 11$ . For easy reference, the term cell shall be used to denote the space in a column with the subheading of a route against a depot. The entry of all the cells  $(i, j)$  depicts the units of distance  $d_{ij}$  from depot  $i$  to the starting point of route  $j$ . The two objective functions of the numerical problem which are sought to be minimized are

$$C = \sum_{l=1}^2 \lambda_{2+l} [k_{2+l} + 11 c_2 + \sum_{j=1}^6 c_{(2+l)j} x_{(2+l)j}] + \sum_{i=1}^2 \sum_{j=1}^6 c_{ij} x_{ij} \quad \dots(14)$$

where

$$c_{ij} = 2 (365) (6) d_{ij} \sum_{t=1}^{20} (1 + 10/100)^{-t} \quad (i = 1, \dots, 2 + 2; j = 1, \dots, 6) \quad \dots(15)$$

and

$$D = \max \{d_{ij} : x_{ij} > 0 \ (i = 1, \dots, 2 + 2; j = 1, \dots, 6)\}. \quad \dots(16)$$

For the 1st two-objective problem of the numerical problem, we find  $q = 6$ . The subsets forming partition of the set  $\{d_{ij} : i = 1, 2, 2 + 1; j = 1, \dots, 6\}$  are as follows:

$$L_1 = \{d_{14}, d_{25}\}, L_2 = \{d_{15}, d_{36}\}, L_3 = \{d_{16}, d_{23}, d_{24}, d_{31}, d_{35}\}$$

$$L_4 = \{d_{11}, d_{22}, d_{26}, d_{34}\}, L_5 = \{d_{12}, d_{21}, d_{33}\}, L_6 = \{d_{13}, d_{32}\}$$

The  $d_{ij}$ 's belonging to  $L_1, L_2, L_3, L_4, L_5, L_6$  have numerical values 5, 4, 3, 2, 1, .5 respectively. The objective function of the 1st single-objective transportation-type problem of the numerical problem is given by

$$Z_1 = \left\{ \begin{aligned} &M_0 (222000 + 74578.8x_{11} + 37289.4x_{12} + 18644.7x_{13} \\ &+ 186447.0x_{14} + 149157.6x_{15} + 111868.2x_{16} + 37289.4x_{21} \\ &+ 74578.8x_{22} + 111868.2x_{23} + 111868.2x_{24} \\ &+ 186447.0x_{25} + 74578.8x_{26} + 111868.2x_{31} + 18644.7x_{32} \\ &+ 37289.4x_{33} + 74578.8x_{34} + 111868.2x_{35} + 149157.6x_{36} \\ &+ M_1 (x_{14} + x_{25}) + M_2 (x_{15} + x_{36}) + M_3 (x_{16} + x_{24} \\ &+ x_{31} + x_{35}) + M_4 (x_{11} + x_{22} + x_{26} + x_{34}) + M_5 (x_{12} \\ &+ x_{21} + x_{33}) + M_6 (x_{13} + x_{32}) \end{aligned} \right\} \quad \dots(17)$$

Applying the standard transportation method and remembering that the priority factors  $M_k$ 's are such that the expression  $\sum_{k=0}^6 L_k M_k$  has the same sign as the nonzero  $L_k$  with the smallest subscript present in it irrespective of the values of other  $L_k$ 's; an optimal basic feasible of the 1st single-objective transportation-type problem is obtain-



ed and is shown in Table II. Entry in the upper half of the cell in Table II depicts the cost associated with the variables. Values of the basic variables of the optimal solution are encircled in the tableau. The optimal value of the objective function  $Z_1$  is given by

$$Z_1 = 3093283.8M_0 + 2M_2 + 5M_3 + 14M_4 + 22M_5 + 8M_6 \quad \dots(18)$$

An optimal basic feasible solution of the 2nd single-objective transportation-type problem of the numerical problem is obtained proceeding exactly in the same way as done to obtain the optimal solution of the 1st single-objective transportation-type problem of the numerical problem. The optimal value of the objective function  $Z_2$  of the 2nd single-objective transportation-type problem is given by

$$Z_2 = 3107094.4M_0 + 6M_3 + 17M_5 + 20M_6 + 8M_7 \quad \dots(19)$$

Now among the two single-objective transportation-type problems of the numerical problem, the objective function of the 1st one has minimum value. So the site 1 is selected for construction of a new depot and the optimal solution of the numerical problem is provided by the optimal solution of its 1st single-objective transportation-type problem. For ready reference, the optimal solution of the numerical problem is shown in Table III.

TABLE III  
*Optimal solution of numerical problem*

Optimal site for new depot	Values of basic variables of optimal solution	Minimal value of expenditure in construction plus present value of expenditure in total dead mileage	Minimum of maximum of dead mileage of individual buses
$I = 1$	$x_{12} = 10, x_{13} = 8, x_{15} = 2, x_{17} = 5, x_{21} = 12, x_{24} = 0, x_{26} = 8, x_{34} = 6, x_{35} = 5$	$C = 3093283.8$	$D = 4$

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## A NOTE ON NORMED NEAR-ALGEBRAS

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In this paper an attempt has been made to extend the theory of Algebras to Normed near-algebras. For this purpose we are led to a definition of a Normed near-algebra which differs from previous definition. A technique has been developed to find the regular elements in the Normed near-algebras and proved that the set of all regular and quasi-regular elements is open. Further it has been shown that the concepts of topological divisors of zero and the spectrum of an element can be carried to Normed near-algebras also.

§1. Near-algebras were studied by Yamamuro<sup>6</sup>, Brown<sup>2</sup> and others (cf. Pilz<sup>3</sup>). In order to extend the theory of Normed algebras to Normed near-algebras we have defined the Normed near-algebra with an additional condition.

*Definition 1*—A (right) near-algebra  $B$  over a field  $F$  is a linear space over  $F$  on which a multiplication is defined such that (i)  $B$  forms a semigroup under multiplication, (ii) multiplication is right distributive with respect to addition, (iii)  $\alpha(xy) = (\alpha x)y$  for all  $x, y \in B$  and  $\alpha \in F$ .

*Definition 2*—A near-algebra  $B$  over the real or complex numbers is called a Normed near-algebra provided that there is associated with each  $x \in B$  a real number  $\|x\|$ , called the norm of  $x$ , with the following properties :

- (i)  $\|x\| \geq 0$  and  $\|x\| = 0$  if and only if  $x = 0$  (the additive identity of  $B$ ),
- (ii)  $\|x + y\| \leq \|x\| + \|y\|$ , for all  $x, y \in B$ ,
- (iii)  $\|\alpha x\| = |\alpha| \|x\|$ ,  $\alpha$  a real or complex number,
- (iv)  $\|xy\| \leq \|x\| \|y\|$ , for all  $x, y \in B$ ,
- (v)  $\|xy - xz\| \leq \|x\| \|y - z\|$ , for all  $x, y, z \in B$ ,
- (vi) If  $B$  has an identity  $e$ , then  $\|e\| = 1$ .

Through out this paper the property (v) plays a crucial role in proving most of the theorems. It follows from the definition (2) that all the operations are continuous.

Through out this paper we assume that  $B$  is a normed near-algebra such that



- (i)  $B$  is complete with respect to the norm defined as above,
- (ii)  $B$  has an identity  $e$ ,
- (iii) For every  $x \in B$ ,  $0.x = x.0 = 0$ .

*Example*— Let  $S$  be a Banach space over the field  $F$  of real or complex numbers. Then the set  $B(S, S)$  of all Lipschitz continuous functions  $x : S \rightarrow S$  such that  $x(0) = 0$ , and having the Lipschitz norm and with the composite map  $x(y(s)) = (xy)(s)$  of  $S$  onto  $S$ , for all  $x, y \in B(S, S)$  and  $s \in S$ , constitute a normed near-algebra<sup>4,5</sup>.

*Definition 3*— An element  $r \in B$  is said to be left (right) regular if there exists an element  $s \in B$  such that  $sr = e$  ( $rs = e$ ).

The element  $s$  is called left (right) inverse for  $r$ . An element which is both left and right regular is called regular element. An element which is not (left, right) regular is called (left, right) singular.

Through out this we denote the set of all regular elements by  $G$  and the set of all singular elements by  $S$ .

The following Lemma is very useful in proving the main theorem.

*Lemma 1*— Let  $B$  be normed near-algebra, then every element  $r \in B$  such that  $\|e - r\| < 1$  is regular.

*PROOF* : Let  $x = e - r$  and  $y_0 = e$ . Define the sequence  $\{y_n\}_{n=1}^{\infty}$  inductively by

$$y_{n+1} = y_0 + xy_n.$$

Then

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|xy_n - xy_{n-1}\| \\ &\leq \|x\| \|y_n - y_{n-1}\| \\ &\leq \|x\|^n \|y_1 - y_0\| \\ &= \|x\|^{n+1} \end{aligned}$$

and hence  $\|y_{n+p} - y_n\| \leq \|x\|^{n+p} + \dots + \|x\|^{n+1}$  for any integer  $p$ .

Therefore  $\{y_n\}$  is a Cauchy's sequence in  $B$  since  $\|x\| < 1$ . As  $B$  is complete, there exists a  $y \in B$  such that  $\lim_{n \rightarrow \infty} y_n = y$ . From the relation  $y_{n+1} = y_0 + xy_n$ , we get  $y = e + xy = e + (e - r)y$ , that is  $ry = e$ .

Next consider

$$\begin{aligned} \|y_n r - e\| &= \|xy_{n-1} r - x\| \\ &\leq \|x\| \|y_{n-1} r - e\| \\ &\leq \|x\|^{n+1}. \end{aligned}$$



Therefore,  $\|yr - e\| = \lim_{n \rightarrow \infty} \|y_n r - e\| = 0$ , that is  $yr = e$ .

Hence  $r$  is regular.

*Remark 1:* Here the inverse of  $r$  is given by  $r^{-1} = \lim_{n \rightarrow \infty} y_n$ , and it can be proved by using the same construction for  $y_n$  that

$$\|r^{-1}\| \leq \frac{1}{1 - \|e - r\|}.$$

*Lemma 2—* Let  $G_l$  denote the set of all left regular elements of  $B$ . If  $x \in G_l$  then any element  $z \in B$  satisfying  $\|x - z\| < \|y\|^{-1}$  (where  $y$  is a left inverse of  $x$ ) belongs to  $G_l$ .

$$\begin{aligned} \text{PROOF: } \|e - yz\| &= \|yx - yz\| \\ &\leq \|y\| \|x - z\| \\ &< 1. \end{aligned}$$

This implies by the Lemma (1) that  $yz \in G$  and hence  $z \in G_l$ , which proves the Lemma.

Similarly we can state :

*Lemma 3—* Let  $G_r$  denote the set of all right regular elements of  $B$ . If  $x \in G_r$  such that the right inverse of  $x$  is  $y$  then any element  $z \in B$  satisfying  $\|x - z\| < \|y\|^{-1}$ , belongs to  $G_r$ .

Now we will prove the main theorem of this paper.

*Theorem 1—* The set of all regular elements  $G$  is an open set in  $B$ .

*PROOF:* Let  $x \in G = G_l \cap G_r$ .

Consider the open ball

$$S(x) = \{z \in B \mid \|z - x\| < 1/\|x^{-1}\|\}.$$

Then for any  $z \in S(x)$ , by Lemma (2) and Lemma (3),  $z$  is regular. Hence  $S(x) \subset G$ . Therefore  $G$  is open.

*Remark 2:* Infact it can be proved that  $G_l$  and  $G_r$  are open and hence  $G = G_l \cap G_r$  is open.

*Theorem 2—* Let  $B$  be the normed near-algebra. The mapping  $r \rightarrow r^{-1}$  is a homeomorphism of  $G$  onto  $G$ .

*PROOF:* It is sufficient to show that the mapping  $r \rightarrow r^{-1}$  is continuous.

Let  $x, r \in G$  such that

$$\|x - r\| < 1/2 \|r^{-1}\|.$$



Then

$$\begin{aligned}\|e - xr^{-1}\| &= \|rr^{-1} - xr^{-1}\| \\ &\leq \|r - x\| \|r^{-1}\| \\ &< 1/2.\end{aligned}$$

Hence  $xr^{-1} \in G$  and  $\|(xr^{-1})^{-1}\| < 2$ , by Remark 1.

Consequently,

$$\begin{aligned}\|x^{-1}\| &= \|r^{-1}rx^{-1}\| \\ &\leq \|r^{-1}\| \|rx^{-1}\| \\ &= \|r^{-1}\| \|(xr^{-1})^{-1}\| \\ &\leq 2\|r^{-1}\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|r^{-1} - x^{-1}\| &\leq \|x^{-1}xr^{-1} - x^{-1}rr^{-1}\| \\ &\leq \|x^{-1}\| \|x - r\| \|r^{-1}\| \\ &\leq 2\|r^{-1}\|^2 \|x - r\|.\end{aligned}$$

This shows that the mapping  $r \rightarrow r^{-1}$  is continuous. Thus we have  $r \rightarrow r^{-1}$  is homeomorphism.

*Remark 3 :* The above theorem implies that the set of all regular elements  $G$  is a topological group.

The following theorem can be proved by using the same techniques as in algebras.

*Theorem 3—* Let  $B$  be a normed near-algebra and  $\{r_n\}$  a sequence of left (right) regular elements of  $B$  which converges to an element  $r \in B$ . If  $s_n$  is a left (right) inverse for  $r_n$  and if  $\{s_n\}$  is a bounded sequence, then  $r$  is also a left (right) regular.

*PROOF :* Let  $\{r_n\}$  be a sequence such that  $r_n \rightarrow r$  in  $B$ . Let  $\{s_n\}$  be a bounded sequence that is  $\sup \|s_n\| = M < \infty$  and  $s_n$  is a left inverse of  $r_n$ .

As  $r_n \rightarrow r$  we get, for  $\epsilon = 1/M > 0$  there exists a number  $n_0 \geq 1$  such that

$$\|r_n - r\| < 1/(M + 1) \text{ for } n \geq n_0.$$

Now consider

$$\begin{aligned}\|e - s_{n_0}r\| &= \|s_{n_0}r_{n_0} - s_{n_0}r\| \\ &\leq \|s_{n_0}\| \|r_{n_0} - r\|\end{aligned}$$

(equation continued on p. 437)



$$\leq \|s_{n_0}\|/(M+1)$$

$$< 1.$$

This implies that  $s_{n_0} r \in G$ , that is  $r \in G_l$ .

§2. *Definition 4*— Let  $B$  be a normed near-algebra. An element  $r \in B$  is said to be quasi-regular if  $e - r$  is left invertible. (cf. Beidleman and Cox<sup>1</sup>).

*Lemma 4*— Let  $B$  be a normed near-algebra, then every element  $x \in B$  such that  $\|x\| < 1$  is quasi-regular.

PROOF: This is obvious, since  $\|x\| < 1$  implies that  $\|e - (e - x)\| < 1$ . Hence by Lemma 1,  $e - x$  is invertible, that is  $x$  is quasi-regular.

*Theorem 4*— Let  $Q$  denote the set of all quasi-regular elements of  $B$ . Then  $Q$  is open in  $B$ .

PROOF: We know that  $G_l$  is an open set in  $B$ . But an element  $r \in Q \Leftrightarrow (e - r) \in G_l$ . Hence  $Q$  is open in  $B$ .

*Definition 5*— An element  $z$  in the normed near-algebra  $B$  is called a left (right) topological divisor of zero provided there exists a sequence  $\{z_n\}$  in  $B$  such that  $\|z_n\| = 1$ , for all  $n$  and  $zz_n \rightarrow 0$  ( $z_n z \rightarrow 0$ ).

We call the element which is either a left or right topological divisor of zero as a topological divisor of zero. We denote this set by  $Z$  and the set of all left (right) topological divisors of zero in  $B$  by  $Z_l$  ( $Z_r$ ).

We now state the following theorems without proofs since they are very easy and can be proved using the same techniques as in algebras.

*Theorem 5*— The set of all topological divisors of zero is a subset of the set of all singular elements. In other words in the normed near-algebra  $B$  every topological divisor of zero is singular.

*Theorem 6*— The boundary of  $S$  in  $B$  is a subset of  $Z$ .

*Theorem 7*—(a) Every element  $z$  for which  $(e - z)$  is a topological divisor of zero is quasi-singular, that is not quasi-regular.

(b)  $G_l \subseteq H_l$ ,  $G_r \subseteq H_r$  and  $G \subseteq H$ , where  $H_l$ ,  $H_r$  and  $H$  are the sets that are complements of the sets  $G_l$ ,  $G_r$  and  $G$  respectively.

(c)  $G = G_l \cap H_r = G_r \cap H_l$ .

(d)  $G_l \cap S_r \subseteq Z_r$ .

*Definition 6*— Let  $B$  a complex normed near-algebra and let  $x$  be any element of  $B$ . Let  $\sigma_0(x)$ , be the set of all non-zero complex numbers such that  $e - x\lambda^{-1}e$  is



singular. Put  $\sigma(x) = \sigma_0(x)$ , if  $x$  is not singular and  $\sigma(x) = \sigma_0(x) \cup \{0\}$ , if  $x$  is singular.

Then  $\sigma(x)$  is defined as the spectrum of  $x$ .

The complement of  $\sigma(x)$  is denoted by  $\rho(x)$ .

**Theorem 8**—  $\sigma(x)$  is bounded and closed in  $B$ .

**PROOF:** Let  $\lambda \in \sigma_0(x)$ . By the Lemma 1, we get  $\|e - (e - x\lambda^{-1}e)\| < 1$ . That is  $\|x\lambda^{-1}e\| < 1$ , which implies that  $\|x\| \geq |\lambda|$ . Hence  $\sigma(x)$  is bounded.

The mapping  $\lambda \rightarrow x\lambda^{-1}e$  is continuous mapping from  $\mathbb{C} - \{0\}$  to  $B$ . Hence Theorem 1, gives that  $\sigma_0(x)$  is closed and hence  $\sigma(x)$  is closed. Thus the proof is complete.

But it is an open problem to settle whether  $\sigma(x)$  is non-empty or not when  $x$  is not singular.

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## RADICAL GOLDIE NEAR-RINGS

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Here we prove the existence of the classical near-ring of right quotients of the sub near-ring  $D$  of the distributive elements of a prime Goldie Abelian near-ring  $K$  which is such that some power of each element of  $K$  belongs to  $D$  and further its non nilpotent elements are distributive.

§1. We have already introduced the notion of a Goldie near-ring and have established some results on prime and semiprime Goldie near-rings, proved Goldie Theorem analogue for Abelian Goldie near-rings in which non nilpotent elements are distributive<sup>1</sup>.

In this paper we discuss a radical Goldie near-ring. If  $A$  is a subnear-ring of a near-ring  $K$  then  $K$  is radical over  $A$  (or  $K$  is  $A$ -radical) if a power of each element in  $K$  lies in  $A$ . In an Abelian near-ring in which non nilpotent elements are distributive<sup>1</sup>  $K$  is  $D$ -radical, where  $D$  is the subnear-ring of distributive elements of  $K$ . Here we establish that if  $K$  is a prime Goldie Abelian near-ring which is radical over the subnear-ring  $D$  of distributive elements of  $K$  and further its non nilpotent elements are distributive, then  $D$  has a classical near-ring of right quotients. Unless otherwise specified a near-ring  $K$  will contain unity and  $a.0 = 0$  for all  $a \in K$ .

§2. A countable ordered family  $\{A_1, A_2, \dots\}$  of subsets of a near-ring  $K$  is an independent family if for all  $n \in \mathbb{N}$ ,  $A_i \cap (\sum_{\substack{k=1 \\ k \neq i}}^n A_k) = 0$ , where  $1 \leq i \leq n$  and  $1 \leq k \leq n$ .

A right (left)  $K$ -subset  $A$  of (right) near-ring  $K$  is a subset of  $K$  such that  $ak (ka) \in A$  for  $a \in A, k \in K$ . If  $A$  is a right and a left  $K$ -subset of  $K$  then  $A$  is a  $K$ -subset of  $K$ .  $A$  is a right (left) essential  $K$ -subset of  $K$  if for any non zero right (left)  $K$ -subset of  $K$  has a nonzero intersection with  $A$ . A near-ring  $K$  is semiprime if  $K$  has no nonzero nilpotent  $K$ -subset,  $K$  is prime if  $0$  is a prime  $K$ -subset (i. e. for any two  $K$ -subsets  $A, B$  of  $K$ ,  $AB = 0 \Rightarrow A = 0$  or  $B = 0$ ).

A Goldie near-ring is a (right) near-ring  $K$  with a.c.c. on right annihilator  $K$ -subsets and which has no infinite independent family of right  $K$ -subsets of it.

In an Abelian near-ring  $K$ , the subset  $D$  of distributive elements of  $K$  is a subnear-ring of  $K$  (it is actually a ring). The following two Lemmas are easy.



*Lemma 2.1.1*— Let  $K$  be an Abelian near ring and  $S$  be a set of distributive elements of  $K$ . Then the right annihilator  $K$ -subset  $r_A(S)$  is a right ideal of  $K$ .

*Lemma 2.1.2*— If  $A$  is a subnear-ring of a near-ring  $K$  and  $S \subseteq A$ , then  $r_A(S) = r_K(S) \cap A$ .

*Lemma 2.2.1*— Let  $K$  be a near-ring. Suppose  $k \in K$  is such an element that for each  $x \in K$  there exists  $n \in \mathbb{Z}^+$  (depending on  $x$ ) satisfying the condition  $kx^n k = 0$ . (In Lemma 2.2.7 we see that such a condition is justified). Then there exists a nonzero element  $a$  of  $K$  with  $a^2 = 0$  and satisfying the same hypothesis as  $k$ .

PROOF: Assume  $x = k$ . Then  $k^\alpha = 0$  for some  $\alpha > 2$ . For  $k^2 = 0$  we get the required result on taking  $k = a$ . And if  $k^2 \neq 0$  let  $\alpha$  be minimal such that  $k^\alpha = 0$ . Then  $a = k^{\alpha-1}$  ( $\neq 0$ ) satisfies the required hypothesis.

*Lemma 2.2.2*— Let  $K$  be a semiprime Goldie near-ring and  $k \in K, k \neq 0$  be as in Lemma 2.2.1.

Then there exists  $a \in K$  with  $a^2 = 0$  such that  $a$  satisfies same hypothesis as  $k$  and for any  $m \in K$  with  $m^2 = 0$  we have  $ama = 0$ .

PROOF: By above Lemma there exists  $a \in K, a \neq 0, a^2 = 0$  satisfying the same hypothesis as  $k$ . Let  $m \in K$  be such that  $m^2 = 0$ . Then by given hypothesis there exists  $\alpha \geq 1$  such that for any  $x \in K$  we have

$$a(m + maxa)^\alpha a = 0, \text{ or } a(m + maxa)^{\alpha-1}(m + maxa)a = 0$$

or,  $a(m + maxa)^{\alpha-1}(ma) = 0$  (since right distributivity holds in  $K$  and  $a^2 = 0$ ). Using the fact that  $m^2 = 0$  and repeating in like way we get finally  $(xama)^\alpha = 0$ .

Thus,  $Kama$  is a nil left  $K$ -subset of  $K$ . And  $K$  being semi-prime we therefore get that  $ama = 0$  ( $1 \in K$ ).

*Lemma 2.2.3*— Let  $K$  be as in Lemma 2.2.2. Then there exists  $a \in K, a \neq 0, a^2 = 0$  such that  $a$  satisfies the same hypothesis as  $k$  in the above Lemma and for any  $p, q \in K, pq = 0$  implies  $paq = 0$ .

PROOF: From above Lemma we get  $a \in K$  such that  $a^2 = 0, a \neq 0$  and  $a$  satisfies same hypothesis as  $k$  and for any  $m \in K$  with  $m^2 = 0$  we have  $ama = 0$ .

Now let  $p, q \in K$  and  $pq = 0$ . Then for any  $x \in K, (qxp)^2 = 0$  which gives  $a(qxp)a = 0$  for any  $x \in K$ . Thus  $(Kpaq)^2 = 0$  from which  $Kpaq = 0$  follows for  $K$  is semiprime Goldie. Therefore  $paq = 0$  (since  $1 \in K$ ).

*Lemma 2.2.4*— Let  $K$  be a semiprime Goldie near-ring and  $k \in K$  be such that for each  $x \in K$ , there exists an  $n \in \mathbb{Z}^+$  (depending on  $x$ ) such that  $kx^n k = 0$ . Then  $k = 0$ .

PROOF: If  $k \neq 0$ , then by Lemma 2.2.3 we get  $a \in K, a \neq 0, a^2 = 0$  such that  $a$  satisfies the hypothesis as in the Lemma. Now for any  $x \in K$  we get  $n \in \mathbb{Z}^+$  such



that  $ax^n a = 0$ . If  $n = 1$  then  $axa = 0$ . Therefore  $(xa)^2 = 0$ . And if  $n > 1$  then  $ax^n a = 0$  gives  $(ax^{n-1}) a (xa) = 0$ . By the hypothesis satisfied by  $a$  we get  $(ax^{n-1}) a (xa) = 0$ . And this gives  $ax^{n-2} (xa)^2 = 0$ . In like manner we get  $ax^{n-3} (xa)^3 = 0$ . Finally we get  $(xa)^{n+1} = 0$ . Thus in any case we get  $(xa)^{n+1} = 0$ . So  $Ka$  is a nilpotent left  $K$ -subset of  $K$ . And  $K$  being semiprime it therefore follows that  $Ka = 0$  which gives  $a = 0$ , a contradiction. Hence  $k = 0$ .

**Lemma 2.2.5**— Let  $K$  be a prime Goldie near-ring. Suppose  $a, b \in K, a \neq 0$  and for each  $x \in K$  there exists an  $n \in \mathbb{Z}^+$  (depending on  $x$ ) such that  $ax^n b = 0$ . Then  $b = 0$ .

**PROOF**: Let  $\rho = \{y \in K \mid y x^{n(x)} b = 0, x \in K\}$ . Since for any  $k \in K$ ,  $k y x^{n(x)} b = 0$  is a left  $K$ -subset of  $K$ . And from Lemma 2.2.4 (by)  $x^{n(x)} (by) = 0$  gives  $by = 0$ . Thus  $b\rho = 0$  which gives  $(KbK)(\rho K) \subseteq Kb\rho K (= 0)$ . And since  $\rho K \neq 0$  and  $K$  is prime it therefore follows that  $KbK = 0$ . And hence  $b = 0$ .

**Lemma 2.2.6**— Let  $K$  be a prime Goldie near-ring and  $A$  be any subnear-ring  $K$ . If  $K$  is  $A$ -radical, then

- (i) for any  $A$ -subset  $I, J$  of  $A$  with  $IJ = 0, I \neq 0$ ; we get  $J = 0$ ;
- (ii) if  $a \in A$  and  $r_A(a) = 0$  then  $r_K(a) = 0$ .

**PROOF**: (i) Let  $a \in I, a \neq 0$  and  $b \in J$ ;  $K$  being  $A$ -radical for  $x \in K$  there exists  $n \in \mathbb{Z}^+$  such that  $x^n \in A$ . Now  $ax^n b \in IAJ \subseteq IJ (= 0)$ . Thus  $ax^n b = 0, a \neq 0$ . Therefore by Lemma 2.2.5 we get  $b = 0$ , So  $J = 0$ .

(ii) If  $r_K(a) \neq 0$ , let  $x \in r_K(a), x \neq 0$ . Then for some  $n \in \mathbb{Z}^+$  we have  $x^n \in A$  and  $ax^n = 0$ . So  $x^n \in r_A(a)$  which means that  $x$  is nilpotent. Thus  $r_K(a)$  is a nil right  $K$ -subset of  $K$ . And  $K$  being prime it therefore follows that  $r_K(a) = 0$ .

**Lemma 2.2.7**— Let  $K$  be a semiprime Goldie near-ring and  $A$  a subnear-ring of  $K$  such that  $K$  is  $A$ -radical. Then

- (i)  $A$  cannot have any nonzero nil right (left)  $A$ -subset;
- (ii) if for  $a_1, a_2 \in A$  we have  $a_1 A \cap a_2 A = 0$ , then  $a_1 K \cap a_2 K = 0$ .

**PROOF**: First we show that  $A$  cannot have nilpotent right  $A$ -subset. Let  $\rho$  be a nilpotent right  $A$ -subset of  $A$ . Without loss of generality we may assume  $\rho^2 = 0$  and let  $a \in \rho, x \in K$ . We get  $n \in \mathbb{Z}^+$  such that  $x^n \in A$ . And then  $ax^n a \in \rho A \rho \subseteq \rho\rho (= 0)$ . Thus for any  $x \in K$  there exists  $n \in \mathbb{Z}^+$  such that  $ax^n a = 0$ . And by Lemma 2.2.4 we get  $a = 0$ . Therefore  $\rho = 0$ . Now we show that  $A$  can not have nonzero nil right  $A$  subset. For this we show that  $a$  satisfies the a.c.c. for right annihilators.

Consider an ascending chain

$r_A(S_1) \subseteq r_A(S_2) \subseteq \dots$  of right annihilator; where  $S_1, S_2, \dots, \subseteq A$ . Then



$r_K(l_A(r_A(S_1))) \subseteq r_K(l_A(r_A(S_2))) \subseteq \dots$ . Since  $K$  is Goldie, we therefore get  $t \in \mathbb{Z}^+$  such that

$$r_K(l_A(r_A(S_t))) = r_K(l_A(r_A(S_{t+1}))) = \dots$$

or,

$$A \cap (r_K(l_A(r_A(S_t)))) = A \cap (r_K(l_A(r_A(S_{t+1})))) = \dots$$

And because of Lemma 2.1.2 we get

$$r_A(l_A(r_A(S_t))) = r_A(l_A(r_A(S_{t+1}))) = \dots$$

Hence

$$r_A(S_t) = r_A(S_{t+1}) = \dots$$

(ii) Consider the right  $A$ -subset  $\rho = a_1 K \cap a_2 A$ . Let  $x \in \rho$ . Then  $x = a_1 k_1 = a_2 k_2$  where  $k_1 \in K, k_2 \in A$ . Since  $K$  is  $A$ -radical there exists  $n \geq 1$  such that  $(k_1 a_1)^n \in A$ . Since  $a_1, k_2 \in A$  we get  $a_1 (k_1 a_1)^n = (a_1 k_1)^n a_1 = (a_2 k_2)^n a_1 \in a_1 A \cap a_2 A (= 0)$ .

Therefore

$$a_1 (k_1 a_1)^n = 0.$$

Hence

$$x^{n+1} = (a_1 k_1) (a_1 k_1)^n = a_1 (k_1 a_1)^n k_1 = 0.$$

Thus  $\rho$  is a nil right  $A$ -subset of  $A$ . Therefore  $\rho = 0$ . So  $a_1 K \cap a_2 A = 0$ .

Now let  $x \in a_1 K \cap a_2 K$  and suppose  $x = a_1 t_1 = a_2 t_2$ , where  $t_1, t_2 \in K$ . Since  $K$  is  $A$ -radical we get  $m \in \mathbb{Z}^+$  such that  $(t_2 a_2)^m \in A$ . Then  $(a_1 t_1)^m a_2 = (a_2 t_2)^m a_2 = a_2 (t_2 a_2)^m \in a_1 K \cap a_2 A (= 0)$ . So  $x^{m+1} = (a_1 t_1)^{m+1} = 0$ . Thus  $a_1 K \cap a_2 K$  is a nil right  $K$ -subset of  $K$ .  $K$  being semiprime, we therefore get  $a_1 K \cap a_2 K = 0$ .

The following Lemmas have been proved in Chowdhury<sup>1</sup>.

**Lemma 2.2.8 :** For any right essential  $K$ -subset  $A$  of a near-ring  $K$ , if  $a \in A$  then

$$a^{-1} A = \{x \in K \mid ax \in A\}$$

is a right essential  $K$ -subset of  $K$  (Corollary 2.3.3 of Chowdhury<sup>1</sup>).

**Lemma 2.2.9—** Let  $K$  be a Goldie near-ring whose non-nilpotent elements are distributive. If  $x \in K$  be such that  $r(x) = 0$ , then  $xK$  is a right essential  $K$ -subset of  $K$  (Lemma 2.4.1 of Chowdhury<sup>1</sup>).

**Lemma 2.2.10—** A semiprime right Goldie near-ring is right non-singular (Lemma 2.4.3 of Chowdhury<sup>1</sup>).

**Lemma 2.2.11—** If  $K$  is a semi prime Goldie near-ring where non nilpotent



elements are distributive then every right essential  $K$ -subset of  $K$  contains a regular element. (Lemma 2.4.4 of Chowdhury<sup>1</sup>).

*Lemma 2.2.12*— Let  $K$  be a semiprime Abelian Goldie near-ring such that its non-nilpotent elements are distributive. Then,  $K$ -satisfies the d.c.c. for right annihilator ideals (Lemma 2.4.5 of Chowdhury<sup>1</sup>).

*Lemma 2.3.1*—Let  $K$  be a prime Goldie Abelian near-ring which is such that its non-nilpotent elements are distributive. If  $D$  is the sub near ring of distributive elements of  $K$ , then  $D$  satisfies the D.C.D. on right annihilators.

PROOF : If  $S \subseteq D$ , then  $r_D(S) = r_K(S) \cap D$ .

Now let  $r_D(S_1) \supseteq r_D(S_2) \supseteq \dots$ , where  $S_1, S_2, \dots \subseteq D$  be a descending chain of right annihilators in  $D$ . Then

$$r_K(l_D(r_D(S_1))) \supseteq r_K(l_D(r_D(S_2))) \supseteq \dots$$

Here  $r_K(l_D(r_D(S_1)))$ , ... etc. are right ideals. Therefore by Lemma 2.2.12, there exists  $t \in \mathbb{Z}^+$  such that

$$r_K(l_D(r_D(S_t))) = r_K(l_D(r_D(S_{t+1}))) = \dots$$

Therefore  $D \cap r_K(l_D(r_D(S_t))) = D \cap r_K(l_D(r_D(S_{t+1}))) = \dots$

And by Lemma 2.1.2 we get  $r_D(l_D(r_D(S_t))) = r_D(l_D(r_D(S_{t+1})))$  which gives  $r_D(S_t) = r_D(S_{t+1}) = \dots$

Thus  $D$  satisfies the D.C.C. on right annihilators.

*Lemma 2.3.2*— Let  $K$  be as above. If for some  $a \in D$ , we have  $r_D(a) \neq 0$  then  $aD$  is not a right essential  $D$ -subset of  $D$ .

PROOF : If  $aK \leq_e K$ , then by Lemma 2.2.11  $r_K(a) = 0$  and therefore  $r_D(a) = r_K(a) \cap D = 0$ , a contradiction. So  $aK \not\leq_e K$ . If possible let  $aD \leq_e D$  and consider any nonzero right  $K$ -subset  $C$  of  $K$ . Then  $C$  is not nil and let  $c$  be a non-nilpotent element of  $C$ . By hypothesis  $c \in D$ . Since  $1 \in D$ ,  $cD \neq 0$ . So  $aD \cap cD \neq 0$ .

Now

$$aK \cap C \supseteq aK \cap cK \supseteq aD \cap cD \neq 0.$$

Thus  $aK \leq_e K$ , a contradiction. Hence  $aD \not\leq_e D$ .

*Lemma 2.3.3*— Let  $K$  and  $D$  be as above and  $K$  is  $D$ -radical. Then a right  $D$ -subgroup  $\lambda$  of  $D$  which is also a right essential  $D$ -subset of  $D$  contain a regular element.

PROOF : Choose  $a \in \lambda$  so that  $r_D(a)$  is minimal in  $D$  (Lemma 2.3.1). We claim that  $r_D(a) = 0$ .



If  $r_D(a) \neq 0$  then by Lemma 2.3.2  $aD \not\leq_e D$ . So there is a nonzero right  $D$ -subset  $J$  of  $D$  such that  $aD \cap J = 0$ . Since  $\lambda \leq_e D$ ,

$$\lambda \cap J \neq 0. \text{ Write } J' = \lambda \cap J.$$

Then

$$aD \cap J = aD \cap (\lambda \cap J) = (aD \cap J) \cap \lambda = 0.$$

If  $x \in J'$  and  $d \in r_D(x + a)$ , then  $xd + ad = 0$

Or,

$$\begin{aligned} ad &= -xd = y(-d) \text{ (since } x \in D) \\ &= 0 \text{ (for } aD \cap J = 0). \end{aligned}$$

Thus  $d \in r_D(a) \cap r_D(x)$ . Therefore  $r_D(x + a) \subseteq r_D(a) \cap r_D(x) \subseteq r_D(a)$ .

And  $x + a \in \lambda$  (since  $\lambda$  is a subgroup of  $D$ ).

Hence we get, by minimality of  $r_D(a)$ ,  $r_D(x + a) = r_D(a)$ .

So  $r_D(x + a) = r_D(a) \cap r_D(x) = r_D(a)$  which gives  $r_D(a) \subseteq r_D(x)$ . Hence  $x r_D(a) = 0$ . Therefore we get  $J' r_D(a) = 0$ . Thus  $(DJ')(Dr_D(a)) \subseteq DJ' r_D(a) (= 0)$ . So  $Dr_D(a) = 0$  for  $DJ' \neq 0$  (since  $1 \in D$ ) (Lemma 2.2.6 (i)). It follows that  $r_D(a) = 0$ . Therefore  $r_K(a) = 0$  (Lemma 2.2.6 (ii)). Now let  $c \in 1_K(a)$ . Then  $ca = 0$  which gives  $caK = 0$ . By Lemma 2.2.9  $aK \leq_e K$  and therefore  $c \in Z(K)$ . And  $K$  being right non singular (Lemma 2.2.10), we get  $c = 0$ . Thus  $1_K(a) = 0$ . It follows that  $a$  is regular in  $D$ .

§3. We now prove the main result.

**Theorem 3.1 :** Let  $K$  be a prime Goldie Abelian near-ring which is  $D$ -radical and whose non nilpotent elements are distributive. Then  $D$  has a classical near-ring of right quotients.

**PROOF :** Let  $a, b \in D$  and  $a$  be regular in  $D$ . Then  $r_D(a) = 0$ . By Lemma 2.2.6 (ii) we get  $r_K(a) = 0$ . So  $aK \leq_e K$  (Lemma 2.2.9). Now if  $aD \leq_e D$ , then there exists a non zero right  $D$ -subset  $\rho$  of  $D$  such that  $aD \cap \rho = 0$ . Let  $x \in \rho$ ,  $x \neq 0$ , we then get  $aD \cap xD \subseteq aD \cap \rho (= 0)$  or  $aD \cap xD = 0$  which gives  $aK \cap xK = 0$  (Lemma 2.2.7 (ii)) put  $aK \leq_e K$  then gives  $xK = 0$  which is a contradiction (since  $1 \in K$ ). So  $aD \leq_e D$ .

Now let  $\lambda = \{x \in D \mid bx \in aD\}$ . Then by Lemma 2.2.8,  $\lambda \leq_e D$ . And  $\lambda$  is a subgroup of  $D$  (Since  $b$  is distributive and  $aD$  is a right ideal of  $D$ ). Thus  $\lambda$  is a right  $D$ -subgroup of  $D$ . So by Lemma 2.3.3,  $\lambda$  contains a regular element, say  $a'$ . Hence  $ba' = ab$ , for some  $b' \in D$ . Thus the right Ore condition is satisfied with respect to the set of regular elements in  $D$ . Therefore by Theorem 3.3<sup>1</sup>  $D$  has a classical near ring of right quotients.

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# SOME RESULTS ON STABILITY OF DIFFERENTIAL SYSTEMS WITH IMPULSIVE PERTURBATIONS

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The purpose of the present work is to derive a few results which improve upon some earlier finding on stability analysis of solutions of ordinary differential systems. Firstly, we treat a most general differential system with impulsive perturbations and study the stability of the solutions of such a system. The results thus derived unify and improve the results of Raghvendra and Rao<sup>4</sup> and Strauss and Yorke<sup>5</sup> simultaneously. Further we also treat the stability problem of a weakly linear differential system which is new and infact is an extension of result of Raghavendra and Rao. One final result is derived on necessary and sufficient condition for stability of specific nonlinear differential system.

## 1. INTRODUCTION

Raghavendra and Rao<sup>4</sup> investigated some stability properties of solutions of ordinary differential systems with respect to impulsive perturbation. In fact slight modification of Gronwall-Bellman type integral inequalities and their extentions, which has been given by them, is the main tool in considering the stability theorem for solutions of ordinary differential equations containing measures.

In the present work we try to deal with most general type of differential system analogous to Strauss and Yorke<sup>5</sup> with a difference that we now introduce impulsive perturbation in the system considered by him. Subsequently the stability theorem thus developed extends in theory as purported by Raghavendra and Rao<sup>4</sup>. The next section is devoted to preliminaries and basic lemmas quoted from Raghavendra and Rao<sup>4</sup> and Strauss and Yorke<sup>5</sup> which we have used later on in the sequel. Subsequent sections deal with the main results of the present work.

## 2. PRELIMINARIES AND BASIC LEMMAS

For any vector  $x \in R^n$ , the euclidean space of dimension  $n$ , let  $|x| = \sum_{i=1}^n |x_i|$ . By  $C[E, R^n]$  we denote the class of continuous mappings from  $E$  into  $R^n$ .

Let  $J = [0, \infty]$ . Most general type of differential system with impulsive perturbation is given as follows.

$$Dx = f(t, x) + g(t, x) Dv + h(t) \quad \dots(2.1)$$

where  $x \in R^n$ ,  $Dv$  denotes the distributional derivative of the function  $v$ ,  $f, g \in C[J \times R^n, R^n]$  and  $v: J \rightarrow R$  is a function of bounded variation and it is right continuous on  $J$ .  $Dv$  can also be identified with a Stieltjes measure and in fact possesses the effect of the sudden change of the state of the system at the points of discontinuity of  $u$ .

Equations (2.1) may be regarded as a perturbed system of ordinary differential equation

$$Dx = f(t, x) \quad \dots(2.2)$$

where the perturbation  $g(t, x) Dv$  is impulsive and  $h(t)$  satisfies certain smoothness condition of the type as discussed in Strauss and Yorke<sup>5</sup>, to be stated later on in the sequel. We also assume that  $g(t, x)$ ,  $f(t, x)$  and  $h(t)$  satisfy certain smoothness conditions sufficient to guarantee the existence and the uniqueness of the solutions of (2.1).

We assume the following conditions :

( $G_1$ ) There exists  $\alpha > 0$  such that if  $|x| \leq \alpha$ , then  $|g(t, x)| \leq \gamma(t)$  for all  $t \geq 0$ , where

$$G(t) = \int_t^{t+1} \gamma(s) dv(s) \rightarrow 0 \text{ as } t \rightarrow \infty \quad \dots(2.3)$$

where  $\gamma \in C[J, R_+]$ .

( $G_2$ ) There exists a continuous, non-increasing function  $H(t)$  satisfying :

$$\lim_{t \rightarrow \infty} H(t) = 0$$

$$\text{such that } \left| \int_{t_0}^t h(s) ds \right| \leq H(t_0) \text{ for every}$$

$$0 \leq t_0 \leq t \leq t_0 + 1.$$

We shall prove that if  $g(t, x)$  satisfies ( $G_1$ ) and  $h(t)$  satisfies ( $G_2$ ) then there exists  $T_0 > 0$  and  $\delta_0 > 0$  such that if  $t_0 \geq T_0$  and  $|x_0| < \delta$ , the solution  $F(t, t_0, x_0)$  of (2.1) approaches zero as  $t \rightarrow \infty$ . In particular if  $x = 0$  is a solution of (2.1) then it is uniformly asymptotically stable. In fact on the other hand if  $g(t, x) = 0$  and  $h(t)$  does not satisfy ( $G_2$ ), then no solution of (2.2) tends to zero as  $t \rightarrow \infty$ .

In the case  $f(t, x) = Ax$ , where all the characteristic roots of  $A$  have negative



real parts, we derive some stability results for the so called weakly nonlinear differential system of the form

$$Dx = Ax + F(t, x) + g(t, x) Dv + h(t) \quad \dots(2.4)$$

we now give some definition in the next paragraph.

*Definition 2.1*—The null solution of (2.1) is said to be uniformly asymptotically stable if the following two conditions hold :

- (i) for every  $\epsilon > 0$ , there exists a  $\delta = \delta(\epsilon) > 0$  and

$$\tau = \tau(\epsilon) > 0 \text{ such that } |y(t, t_0, x_0)| < \epsilon,$$

$$t \geq t_0 \geq \tau(\epsilon) \text{ provided that } |x_0| < \delta.$$

- (ii) for every  $\eta > 0$ , there exists positive numbers

$$\delta_0, \tau_0 \text{ and } T = T(\eta) \text{ such that } |y(t, t_0, x_0)| < \eta$$

$$t \geq t_0 + T, t_0 > \tau_0 \text{ provided } |x_0| < \tau_0.$$

In the next section we would extend the proof of the Theorem 3.1 of Raghavendra and Rao<sup>4</sup> to get our result and the proof thus given does not make use of the Lyapunov function as also done in the case of Theorem 5.1 of Strauss and Yorke<sup>5</sup>. Before we prove our result we give some lemmas which are basically the extensions of the lemmas which appear in Strauss and Yorke<sup>5</sup> and we also state the Gronwall type of inequalities quoted from Strauss and Yorke<sup>5</sup> which has been used further in our result. We also remark here that the generalized inequality of Raghavendra and Rao<sup>4</sup> is not needed in the proof of our result, instead the more traditional inequality such as the one given in Lemma 2.4 below suffices for the derivation of Theorem 3.1.

$$\text{Lemma 2.1—} \int_{t_0}^t \gamma(s) dv(s) \leq \int_{t_0-1}^t G(s) ds \text{ for all } t \geq t_0 \geq 1.$$

$$\text{Lemma 2.2—} \int_{t_0}^t e^{\sigma s} \gamma(s) dv(s) \leq \int_{t_0-1}^t e^{\sigma(s+1)} G(s) ds \text{ for all } \sigma > 0, t \geq t_0 \geq 1.$$

$$\text{Lemma 2.3—} \lim_{t \rightarrow \infty} e^{-\sigma t} \int_1^t e^{\sigma s} \gamma(s) dv(s) = 0 \text{ for all } \sigma > 0.$$

*Lemma 2.4*—Let  $\gamma(t)$  and  $p(t)$  be non-negative and continuous for  $t \geq t_0$ , let  $C \geq 0$ ,  $k \geq 0$ , and let

$$\gamma(t) \leq C + \int_{t_0}^t [k \gamma(s) + p(s)] ds$$

then

$$\gamma(t) \leq C e^{k(t-t_0)} + \int_{t_0}^t p(s) e^{k(t-s)} ds.$$

The following theorem gives the sufficient conditions for uniform asymptotic stability of (2.1) with respect to (2.2). Let  $f(t, 0) = 0$  for all  $t \geq 0$ .

**Theorem 3.1**—Let the null solution of (2.2) be uniformly asymptotically stable. Assume that :

(i)  $f$  satisfies the uniform Lipschitz continuous condition of the following type.

$$|f(t, x) - f(t, y)| \leq L |x - y|, \quad |x|, |y| \leq a$$

where  $a$  is a positive real number.

(ii)  $g(t, x)$  satisfies the condition as in  $(G_1)$  and (2.3) holds.

Then there exists  $T_0 \geq 0$  and  $\delta_0 > 0$  such that  $t_0 \geq T_0$  and  $|x_0| < \delta_0$ , the solution  $y(t, t_0, x_0)$  of (2.1) satisfies  $|y(t, t_0, x_0)| \rightarrow 0$  as  $t \rightarrow \infty$ . In particular if  $g(t, 0) = 0$  then the null solution of (2.1) is eventually asymptotically stable.

**PROOF** : The solutions and constants corresponding to system (2.2) shall be starred, those for (2.1) shall not. Now let

$$Q(t) = \sup \{G(s) : t-1 \leq s < \infty\}.$$

Then it is easy to see that

$$Q(t) \downarrow 0 \text{ as } t \rightarrow \infty.$$

Invoking lemma 2.1 we get

$$\int_{t_0}^t \gamma(s) dv(s) \leq \int_{t_0-1}^t G(s) ds \leq Q(t_0)(t - t_0 + 1)$$

if  $t \geq t_0 \geq 1$ . Also we have from condition  $(G_2)$

$$\begin{aligned} \left| \int_{t_0}^t h(s) ds \right| &\leq \left| \int_{t_0}^{t_0+1} h(s) ds \right| + \dots + \left| \int_{t_0+m}^t h(s) ds \right| \\ &\leq H(t_0) \dots \dots \dots + H(t_0 + m) \\ &\leq H(t_0)(t - t_0 + 1). \end{aligned}$$

Put  $B(t) = Q(t) + H(t)$ . Let  $t_0 \geq 1$  and  $|x_0| \leq \gamma$ .

If  $y(t, t_0, x_0)$  is a solution of (2.1) and if

$|y(t, t_0, x_0)| \leq \gamma$  on  $[t_0, t_0 + \tau]$  for some  $\gamma > 0$ , then

$$|y(t, t_0, x_0) - y^*(t, t_0, x_0)|$$

$$= |x_0 + \int_{t_0}^t f(s, y(s, t_0, x_0)) ds + \int_{t_0}^t g(s, y(s, t_0, x_0)) dv(s)|$$

(equation continued on p. 450)



$$\begin{aligned}
& + \left| \int_{t_0}^t h(s) ds - x_0 - \int_{t_0}^t f(s, y^*(s, t_0, x_0)) ds \right| \\
& \leq \int_{t_0}^t L |y(s) - y^*(s)| ds + \int_{t_0}^t \gamma(s) dv(s) + \left| \int_{t_0}^t h(s) ds \right| \\
& \leq \int_{t_0}^t L |y(s) - y^*(s)| ds + B(t_0)(t - t_0 + 1).
\end{aligned}$$

From Lemma 2.4,

$$\begin{aligned}
& |y(t, t_0, x_0) - y^*(t, t_0, x_0)| \\
& \leq B(t_0) e^{L(t-t_0)} + \int_{t_0}^t B(t_0) e^{L(t-s)} ds \\
& \leq B(t_0) e^{L\tau} + B(t_0) e^{L\tau} (t - t_0) \\
& \leq e^{L\tau} (1 + \tau) B(t_0).
\end{aligned}$$

We may assume without loss of generality that  $\gamma \leq \delta_0^*$ . Next we proceed to show by the estimates as derived earlier<sup>4,5</sup> that  $|y(t, t_0, x_0)| < \epsilon$  on every interval  $[t_0 + m\tau, t_0 + (m+1)\tau]$  and hence on  $[t_0, \infty)$ . Hence if  $g(t, 0) = 0$  and  $h(t) = 0$ , then it is clear that  $x = 0$  is uniformly asymptotically stable. Rest of the proof is completed following Raghavendra and Rao<sup>4</sup>. In fact, given  $0 < \eta < \gamma$  and choosing  $\delta(\eta) = \delta^*(\eta/2)$ ,  $0 < \delta < \eta$ ,

$\tau(\eta) = T^*(\delta/2)$  and  $T_1(\eta)$  so that

$$B(T_1) < \delta [e^L (1 + \tau) L]^{-1}$$

we finally show that  $|y(t, t_0, x_0)| < \eta$ , for all  $t > t_0 + T$ ,

where  $T = T(\eta) = \tau + T_1$ .

#### 4. WEAKLY NON-LINEAR DIFFERENTIAL SYSTEM

We consider a weakly linear differential system with impulsive perturbation in the present section. The results on stability derived subsequently generalizes Theorem 4.1 of Raghavendra and Rao<sup>4</sup>. We consider the following ordinary differential system.

$$Dx = Ax + F(t, x) + g(t, x) Dv + h(t) \quad \dots(4.1)$$

where  $A$  is  $n \times n$  constant matrix and  $F \in C[J \times R^n, R^n]$ . The solution  $y(t)$  of (4.1) satisfying  $y(t_0) = x_0$ ,  $t_0 \in J$ , is given by

$$y(t) = \Phi(t - t_0) x_0 + \int_{t_0}^t \Phi(t - s) F(s, y(s)) ds$$

(equation continued on p. 451)

$$\begin{aligned}
& + \int_{t_0}^t \Phi(t-s) g(s, y(s)) dv(s) \\
& + \int_{t_0}^t \Phi(t-s) h(s) ds
\end{aligned} \quad \dots(4.2)$$

where  $\Phi(t)$  is the fundamental matrix of the equation  $x' = Ax$  satisfying  $\Phi(0) = I$  and  $\Phi \in C^\infty(J)$  where  $I$  denotes the unit matrix<sup>4</sup>.

**Theorem 4.1**—Suppose that

- (i) all characteristic roots of  $A$  have negative real parts
- (ii) given  $\epsilon > 0$ , there exists  $\delta(\epsilon)$ ,  $T(\epsilon) > 0$  such that

$$|F(t, x)| \leq \epsilon |x| \text{ provided } |x| < \delta(\epsilon) \text{ and } t \geq T(\epsilon)$$

- (iii) The condition  $(G_1)$  and  $(G_2)$  of section 2 hold.

Then, there exists  $T_0 \geq 0$  and  $\delta_0 > 0$  such that for every  $t_0 \geq T_0$  and  $|x_0| < \delta_0$ , any solution  $y(t) = y(t, t_0, x_0)$  of (4.1) satisfies

$$\lim_{t \rightarrow \infty} y(t) = 0.$$

If, in particular (4.1) possesses null solution then the null solution is uniformly asymptotically stable.

**PROOF :** We give a brief outline of the proof. Also note that the technique is same as in Raghavendra and Rao<sup>4</sup>. Because all characteristic roots of  $A$  have negative real parts, there exists positive constants  $a$  and  $c_1$ , such that

$$|\Phi(t)| = |e^{At}| \leq c_1 e^{-at} \text{ for } t \geq 0. \quad \dots(4.3)$$

Assume that

$$\delta = \delta((a/2) c_1^{-1}) < \gamma$$

choose  $t_0$  and  $x_0, s_0$  that  $t_0 \geq T = T((a/2) C_1^{-1})$  and  $|x_0| < \delta_0 < \delta$ . So far as  $|y(t)| = |y(t, t_0, x_0)| \leq \delta$ , for  $t > t_0$  we have

$$\begin{aligned}
|y(t)| & \leq c_1 e^{-a(t-t_0)} |x_0| + \int_{t_0}^t (a/2) e^{-a(t-s)} |y(s)| ds \\
& + \int_{t_0}^t c_1 e^{-a(t-s)} \gamma(s) dv(s) + \left| \int_{t_0}^t e^{-a(t-s)} h(s) ds \right|
\end{aligned}$$

which implies that

$$|y(t)| e^{at} \leq c_1 e^{at_0} |x_0| + a/2 \int_{t_0}^t e^{as} |y(s)| ds$$

(equation continued on p. 452)



$$+ \int_{t_0}^t c_1 e^{as} \gamma(s) dv(s) + \int_{t_0}^t e^{as} |h(s)| ds. \quad \dots(4.4)$$

Applying Lemma 2.4, we have from (4.4)

$$\begin{aligned} |y(t) e^{at}| &< c_1 e^{at_0} |x_0| e^{a/2(t-t_0)} + \int_{t_0}^t c_1 e^{as} e^{a/2(t-s)} \gamma(s) dv(s) \\ &+ \int_{t_0}^t e^{a/2(t-s)} e^{as} |h(s)| ds. \end{aligned}$$

From which it follows that

$$\begin{aligned} |y(t)| &\leq c_1 e^{-a/2(t-t_0)} |x_0| + c_1 \int_{t_0}^t e^{-a/2(t-s)} \gamma(s) dv(s) \\ &+ \int_{t_0}^t e^{-a/2(t-s)} |h(s)| ds. \end{aligned}$$

By choosing  $\delta_0$  and  $T_0$  suitably and using the properties of the function  $Q(t)$  and  $H(t)$  we can proceed as Raghavendra and Rao<sup>4</sup> to show that for  $t \geq T_0$  and  $|x_0| \leq \delta$ , we have  $|y(t)| < \delta$ , which implies the existence of  $y(t, t_0, x_0)$  on the interval  $[t_0, \infty)$ . For  $t > T_0 \geq 1$ , the fact that

$$\int_{t_0}^t e^{-a/2(t-s)} \gamma(s) dv(s) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

follows by using the same arguments as in Raghavendra and Rao<sup>4</sup> by using property  $(G_1)$ . We next show that

$$\int_{t_0}^t e^{-a/2(t-s)} |h(s)| ds \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

we have

$$\begin{aligned} \int_{t_0}^t e^{-a/2(t-s)} |h(s)| ds &= \int_{t_0}^{t/2} e^{-a/2(t-s)} |h(s)| ds \\ &+ \int_{t/2}^t e^{-a/2(t-s)} |h(s)| ds \\ &\leq |h(0)| e^{-a/2t} \int_0^{t/2} e^{-a/2s} ds + h(t/2) \int_{t/2}^t e^{-a/2(t-s)} |ds| \end{aligned}$$

from which it is easy to see that the right hand side of the above inequality goes to zero as  $t \rightarrow \infty$ .

Hence

$$|y(t, t_0, x_0)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This also proves that if  $g(t, 0) = 0$  and  $h(t) = 0$ , then the null solution of (4.1) is eventually uniformly stable, which completes the proof.

## 5. A NEW RESULT

In the present section we give a new result on necessary and sufficient conditions on stability of a non-linear differential system with impulsive perturbation.

*Theorem 5.1*—Let  $x = 0$  be uniformly asymptotically stable. Let the condition (i) be satisfied for  $f(t, x)$  as in Theorem 3.1. Also let  $g(t, x)$  satisfy the condition  $(G_1)$ . Then for the following system

$$Gx = f(t, x) + g(t, x) Dv + h(t) \quad \dots(5.1)$$

the conclusions of Theorem 3.1 hold if and only if  $h(t)$  satisfies condition  $(G_2)$ .

**PROOF :** Let  $h$  satisfy condition  $(G_2)$ , then the proof of the Theorem 3.1 establishes this one way conclusion.

Conversely, suppose  $h$  does not satisfy the condition  $(G_2)$ . Then there exist  $\eta > 0$  and sequences  $\{t_n\}$  and  $\{\theta_n\}$  with  $0 \leq \theta_n \leq 1$  for every  $n$  and  $t_n \rightarrow \infty$  as  $h \rightarrow \infty$ , such that

$$\left| \int_{t_n}^{t_n + \theta_n} h(t) dt \right| > \eta \quad \dots(5.2)$$

for every  $n$ . Suppose that there exist some  $t_0 \geq 0$  and some  $x_0$  for which the solution  $y(t, t_0, x_0)$  of (5.1) satisfies  $|y(t, t_0, x_0)| \rightarrow 0$  as  $t \rightarrow \infty$ . Choose  $T$  so large that  $t \geq T$  implies

$$|y(t, t_0, x_0)| < \eta [L]^{-1}$$

where  $L$  is the Lipschitz constant as in condition (1) of Theorem 3.1, we may assume without loss of generality that  $L \geq 1$ . Also from condition  $(G_1)$  it follows, choosing  $n$  large enough that we can get

$$\int_{t_n}^{t_n + \theta_n} \gamma(t) dv(t) < \frac{\eta}{4}$$

choose  $n$  so large that  $t_n \geq T$ , then we have

$$\left| \int_{t_n}^{t_n + \theta_n} h(t) dt \right| = \int_{t_n}^{t_n + \theta_n} f(t, y(t, t_0, x_0)) dt$$

(equation continued on p. 454)



$$\begin{aligned}
& + \int_{t_n}^{t_n+\theta_n} g(t, y(t, t_0, x_0)) dv(t) \\
& - \int_{t_n}^{t_n+\theta_n} y'(t, t_0, x_0) dt \mid \\
\leq & \int_{t_n}^{t_n+\theta_n} L |y(t, t_0, x_0)| dt + \int_{t_n}^{t_n+\theta_n} \gamma(t) dv(t) \\
& + |y(t_n + \theta_n, t_n, x_0)| + |y(t_n, t_0, x_0)| \\
\leq & \eta/4 + \eta/4 + \eta/4 + \eta/4 = \eta
\end{aligned}$$

which contradicts condition (5.2) hence  $|y(t, t_0, x_0)|$  can not tend to zero as  $t \rightarrow \infty$ .  
Completing the proof of the theorem.

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# A TRANSFORMATION OF THE FINSLER METRIC BY AN $h$ -VECTOR

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The propose of the present paper is to find the relation between  $\nu$ -curvature tensors with respect to  $C\Gamma$  of the Finsler spaces  $(M^n, L)$  and  $(M^n, L^*)$  where  $L^*(x, y)$  is obtained from  $L(x, y)$  by the transformation.

$$L^{*2}(x, y) = L^*(x, p) + (X_i, Y^i)^2$$

where  $X_i(x, y)$  is an  $h$ -vector in  $(M^n, L)$ . The relation in  $n$ -fundamental forms of tangent Riemannian hypersurfaces of  $(M^n, L)$  and  $(M^n, L^*)$  have also been obtained.

## 1. INTRODUCTION

Let  $F^n = (M^n, L)$  be an in  $n$ -dimensional Finsler space, where  $M^n$  is an  $n$ -dimensional differentiable manifold and  $L(x, y)$  is the Finsler fundamental function. Matsumoto<sup>7</sup> introduced the transformations of Finsler metric :

$$\bar{L}(x, y) = L(x, y) + X_i y^i \quad \dots(1.1)$$

$$L^{*2}(x, y) = L^2(x, y) + (X_i y^i)^2 \quad \dots(1.2)$$

and obtained the relation between the imbedding class numbers of tangent Riemannian spaces to  $(M^n, L)$ ,  $(M^n, \bar{L})$  and  $(M^n, L^*)$ . It has been assumed that the functions  $X_i$  in (1.1) and (1.2) are functions of coordinate only. Since a concurrent vector field is a function of coordinate only, assuming  $X_i$  as a concurrent vector field, Matsumoto<sup>4</sup> has studied the  $R3$ -likeness of Finsler spaces  $(M^*, L)$  and  $(M^n, \bar{L})$ , Singh and Prasad<sup>14</sup> and Prasad *et al*<sup>10</sup> generalized the concept of concurrent vector field and introduced the semi parallel and concircular vector fields which are functions of coordinate only. Assuming  $X_i$  as a concircular vector field, Prasad, Singh and Singh<sup>10</sup> has studicd the  $R3$ -likeness of  $(M^n, L)$  and  $(M^n, \bar{L})$ .

If  $L(x, y)$  is a metric function of Riemannian space, than  $\bar{L}(x, y)$  reduces to the metric function of Rander's space. Such a Finsler metric was first introduced by



Rander's<sup>12</sup> from the stand point of general theory of relativity and applied to the theory of electron microscope by ingraden<sup>2</sup> who first named it a Rander's space. The geometrical properties of the space have been studied by various authors. Numata<sup>9</sup> studied the properties of  $(M^n, \bar{L})$  which is obtained from Minkowskian space  $(M^n, L)$  by the transformation (1.1). In all these works the functions  $X_i$  are assumed to be a function of coordinate only.

Izumi<sup>1</sup> while studying the conformal transformation of Finsler spaces, introduced the  $h$ -vector  $X_i$  which is  $\nu$ -covariant with respect to Cartan's connection  $C \Gamma$  and satisfies  $L C_{ij}^h X_h = \rho h_{ij}$ . Thus the  $h$ -vector  $X_i$  is not only a function of coordinate but it is a function of direction  $n/4 = \eta$  satisfying  $L \partial_i X_i = \rho h_{ij}$ , Prasad *et al.*<sup>11</sup> has obtained the relation  $|y(t, t_0)$  in imbedding class numbers of  $(M^n, L)$  and  $(M^n, \bar{L})$  where  $\bar{L}(x, y)$  is obtained from  $L(x, y)$  by the transformation (1.1) under the assumption that  $X_i$  is an  $h$ -vector in  $(M^n, L)$ .

## 2. THE FINSLER SPACE $(M^n, L^*)$

Let  $X_i$  be a vector field in the Finsler space  $(M^n, L)$ . If  $X_i$  satisfy the conditions

$$X_i |_{,j} = 0 \quad \dots(2.1)$$

$$L C_{ij}^h X_h = \rho h_{ij} \quad \dots(2.2)$$

then the vector field  $X_i$  is called an  $h$ -vector<sup>1</sup>. Here  $|_{,j}$  denote the  $\nu$ -covariant differentiation with respect to Cartan connection  $C \Gamma$ ,  $C_{ij}^h$  is the Cartan's  $C$ -tensor,  $h_{ij}$  is the angular metric tensor and  $\rho$  is function given by

$$\rho = \frac{1}{n-1} L C^i X_i \quad \dots(2.3)$$

where  $C^i$  is torsion vector  $C_{jk}^i g^{jk}$ . The first of the following lemmas has been proved in<sup>1</sup> while the other two is a direction consequence of the definition of  $h$ -vector.

**Lemma 2.1**—If  $X_i$  is an  $h$ -vector than the function  $\rho$  and  $X_i^* = X_i - \rho l_i$  are independent functions of  $y$ .

**Lemma 2.2**—The magnitude  $X$  of an  $h$ -vector  $X_i$  is independent function of  $y$ .

**Lemma 2.3**—For an  $h$ -vector  $X_i$  we have  $S h_{ijk} X^h = 0$  where  $S h_{ijk}$  is  $\nu$ -curvature tensor of Cartan's connection  $C \Gamma$ .

Let  $X_i$  is an  $h$ -vector in the Finsler space  $(M^n, L)$  and  $(M^n, L^*)$  be another Finsler space whose fundamental metric function  $L^*(x, y)$  is defined by

$$L^{*2}(x, y) = L^2(x, y) + \beta^2(x, y) \quad \dots(2.4)$$

where  $\beta(x, y) = X_i y^i$ . Since  $X_i$  is  $h$ -vector from (2.1) and (2.2) we get

$$\dot{\partial}_j X_i = L^{-1} \rho h_{ij}$$

which after using the indicatory property of  $h_{ij}$   $\dot{\partial}_j \beta = X_j$ . Thus differentiation of (2.4) with respect to  $y^i$  gives

$$L^* l_i^* = L h_i + \beta X_i \quad \dots(2.5)$$

where  $l_i^* = \dot{\partial}_i L^*$  is the normalized element of support in  $(M^n, L^*)$ . The quantities of  $(M^n, L^*)$  will be denoted by star letters

Since  $\partial_j l_j = L^{-1} h_{ij}$ , differentiation of (2.5) with  $y^j$  and application of (2.4) give

$$h_{ij}^* + l_i^* l_j^* = \sigma h_{ij} + l_i l_j + X_i X_j \quad \dots(2.6)$$

where

$$\sigma = \left( 1 + \frac{\beta \rho}{L} \right). \quad \dots(2.7)$$

Hence we have

$$g_{ij}^* = g_{ij} + (1 - \sigma) l_i l_j + X_i X_j. \quad \dots(2.8)$$

From (2.8), the relation between the contravariant components of the fundamental metric tensors can be derived as follows :

$$\begin{aligned} g^{*ij} = \sigma^{-1} g^{ij} - \frac{(1 - \sigma)\beta}{L \lambda} (l^i X^j + l^j X^i) + \frac{(1 - \sigma)(X^2 + \sigma)}{\lambda} l^i l^j \\ + \frac{1}{\lambda} X^i X^j \end{aligned} \quad \dots(2.9)$$

where

$$\lambda = \left\{ \frac{\sigma(1 - \sigma)\beta^2}{L^2} - X^2 - \sigma \right\} \quad \dots(2.10)$$

and  $X$  is the magnitude of the vector  $X^i (= g^{ij} X_j)$ .

From the Lemma 2.1 and relation (2.7) we get

$$\dot{\partial}_i \sigma = \frac{\rho}{L} m_i \quad \dots(2.11)$$

where

$$m_i = X_i - \frac{\beta}{L} h_i. \quad \dots(2.12)$$



Since  $\partial_j l_i = L^{-1} h_{ij}$ , differentiating (2.8) with respect to  $y^k$  and using (2.4), (2.11) and (2.12) we get

$$C_{ijk}^* = \sigma C_{ijk} + \frac{\rho}{2L} (h_{ij} m_k + h_{ki} m_j + h_{jk} m_i). \quad \dots(2.13)$$

From the definition of  $m_i$ , it is evident that

$$\begin{aligned} (a) \quad m_i l^i &= 0 \\ (b) \quad m_i X^i &= m_i m^i = X^2 - \frac{\beta^2}{L^2} \\ (c) \quad h_{ij} m^i &= h_{ij} X^i = m_j \\ (d) \quad C_{ij}^h m^h &= L^{-1} \rho h_{ij}. \end{aligned} \quad \dots(2.14)$$

From (2.9), (2.2), (2.13) and (2.14) we get

$$\begin{aligned} C_{ij}^{*h} &= C_{ij}^h + \frac{\rho}{2L\sigma} \left( h_{ij} m^h + h_j^h m_i + h_i^h m_j \right) \\ &\quad - \frac{(1-\sigma)\beta\rho}{L^2\lambda} \left[ \left\{ \sigma + \frac{1}{2} \left( X^2 - \frac{\beta^2}{L^2} \right) \right\} h_{ij} l^h + m_i m_j l^h \right] \\ &\quad + \frac{\rho}{L\lambda} \left[ \left\{ \sigma + \frac{1}{2} \left( X^2 - \frac{\beta^2}{L^2} \right) \right\} h_{ij} X^h + m_i m_j X^h \right]. \end{aligned} \quad \dots(2.15)$$

We shall now find the  $\nu$ -curvature tensor  $S_{hijk}^*$  of  $(M^n, L^*)$  which with respect to the Cartan's connection  $C \Gamma$ , is defined as

$$S_{hijk}^* = C_{hkm}^* C_{ij}^{*m} - C_{hjm}^* C_{ik}^{*m}. \quad \dots(2.16)$$

Firstly from (2.13) and (2.15) we have

$$\begin{aligned} C_{hkm}^* C_{ij}^{*m} &= C_{hkm} C_{ij}^m + \alpha_1 h_{ij} h_{hk} + \frac{\rho}{2L} (C_{ijk} m_h \\ &\quad + C_{ijh} m_k + C_{ihk} m_j + C_{jhk} m_i) \\ &\quad + h_{hk} m_i m_j \left\{ \frac{\rho^2 \sigma}{2L^2 \lambda} + \frac{\rho^2}{2L^2 \sigma} + \frac{\rho^2}{2L^2 \lambda} \right. \\ &\quad \times \left( X^2 - \frac{\beta^2}{L^2} \right) \left. \right\} + h_{ij} m_h m_k \left[ \frac{\rho^2}{2L^2 \sigma} + \frac{\rho^2}{2L^2 \lambda} \left\{ \sigma \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \left( X^2 - \frac{\beta^2}{L^2} \right) \right\} \right] + \frac{\rho^2}{4L^2 \lambda} (h_{jh} m_i m_k + h_{ih} m_j \\ &\quad + h_{ih} m_j m_k + h_{jk} m_i m_h) \end{aligned}$$

(equation continued on p. 459)

$$\times m_k + h_{jr} m_i m_h + h_{ik} m_j m_h) + \frac{\rho^2}{L^2 \lambda} m_i m_j m_h m_k \quad \dots(2.17)$$

where

$$\begin{aligned} \alpha_1 = & \frac{\rho^2}{L^2} + \frac{\rho^2 \sigma^2}{L^2 \lambda} + \frac{\rho^2 \sigma}{L^2 \lambda} \left( X^i - \frac{\beta^2}{L^2} \right) + \frac{\rho^2}{4 L^2 \sigma} \left( X^2 - \frac{\beta^2}{L^2} \right) \\ & + \frac{\rho^2}{4 L^2 \lambda} \left( X^2 - \frac{\beta^2}{L^2} \right)^2. \end{aligned} \quad \dots(2.18)$$

Thus from (2.16) we get

$$S_{hijk}^* = \sigma S_{hijk} + h_{ij} d_{hk} + h_{hk} + h_{hk} d_{ij} - h_{ik} d_{hj} - h_{hj} d_{ik} \quad \dots(2.19)$$

where

$$d_{ij} = \frac{1}{2} \alpha_1 h_{ij} + \alpha_2 m_i m_j \quad \dots(2.20)$$

$$\alpha_2 = \frac{\rho^2 \sigma}{L^2 \lambda} + \frac{\rho^2}{2 L^2 \lambda} + \frac{\rho^2}{2 L^2 \lambda} \left( X^2 - \frac{\beta^2}{L^2} \right) - \frac{\rho^2}{4 L^2 \lambda} \quad \dots(2.21)$$

### 3. HYPERSURFACE OF $(M^n, L)$

Let  $(M^{n-1}, L)$  be a hypersurface of  $(M^n, L)$  given by the equation

$$x^i = x^i(u^\alpha). \quad \dots(3.1)$$

Let us suppose that the functions (3.1) are atleast of class  $C^3$  in  $u^\alpha$  and the projection factors  $\beta_\alpha^j = \frac{\partial x^j}{\partial u^\alpha}$  are such that their matrix has maximal rank  $n - 1$ . The fundamental metric function  $L(u, v)$  of the hypersurface is given by

$$L(u^\alpha, v^\alpha) = L(x^i(u^\alpha) B_\alpha^i v^\alpha)$$

where  $v^\alpha$  is the element of support for the hypersurface for which

$$y^i = B_\alpha^i v^\alpha.$$

Thus if  $l^\alpha$  denote the normalized vector using the element of support then

$$l^i = B_\alpha^i l^\alpha. \quad \dots(3.2)$$

If  $g_{hj}(x, y)$  denotes the metric tensor of  $(M^n, L)$ , the induced metric tensor of  $(M^{n-1}, L)$  is given by

$$g_{\alpha\beta}(u, v) = g_{hj}(x, y) B_\alpha^h B_\beta^j. \quad \dots(3.3)$$



The inverse of (3.3) is denoted by  $g^{\alpha\beta}(u, v)$  by means of which we define the quantities

$$B_i^\alpha(n, v) = g^{\alpha\beta}(u, v) g_{ij}(x, y) B_\beta^j. \quad \dots(3.4)$$

The unit normal vector  $N^j(x, y)$  of  $(M^{n-1}, L)$  is determined by the relations

$$g_{hj}(x, y) B^h_\beta N^j(x, y) = 0, \quad g_{hj}(x, y) N^h(x, y) N^j(x, y) = 1. \quad \dots(3.5)$$

We have the following identity from (3.3), (3.4) and (3.5)

$$B_j^\alpha B_\beta^j = \delta_\beta^\alpha, \quad B_\alpha^j B_h^\alpha + N^j N_h = \delta_h^j \quad \dots(3.6)$$

where

$$N_i = g_{ij}(x, y) N^j.$$

If  $C_{hjk}(u, y)$  denotes the  $(h)$   $h\nu$ -torsion tensor of  $(M^n, L)$  the induced  $(h)$   $h\nu$ -torsion tensor  $C_{\alpha\beta\gamma}(u, v)$  of  $(M^{n-1}, L)$  is given by

$$C_{\alpha\beta\gamma}(u, v) = C_{hjk}(x, y) B_\alpha^h B_\beta^j B_\gamma^k \quad \dots(3.7)$$

from which we obtain

$$C_{\beta\gamma}^\alpha = B_i^\alpha C_{j k}^i B_\beta^j B_\gamma^k. \quad \dots(3.8)$$

The relative  $\nu$ -covariant derivative of the projection factor  $B_\beta^i$  with respect to the induced Cartan connection  $I C \Gamma$  is defined as<sup>6</sup>

$$B_\beta^i |_\gamma = -B_\alpha^i C_{\beta\gamma}^\alpha + C_{hk}^i B_\beta^h B_\gamma^k. \quad \dots(3.9)$$

This tensor is normal to  $(M^{n-1}, L)$ . Therefore we may write

$$B_{\beta|\gamma}^i = M_{\beta\gamma} N^i. \quad \dots(3.10)$$

From (3.9), it is clear that  $M_{\beta\gamma}$  is symmetric in  $\beta$  and  $\gamma$  and it may be written as

$$M_{\beta\gamma} = C_{ijk} N^i B_\beta^j B_\gamma^k. \quad \dots(3.11)$$

The tangent vector-space  $M_x^{n-1}$  to  $M^{n-1}$  at every point  $x^i (= u^\alpha)$  of the hypersurface is considered as the Riemannian space  $(M_x^{n-1}, g_x)$  with the Riemannian metric  $g_x = g_{\alpha\beta}(u, v) dv^\alpha dv^\beta$ . The components of the  $(h)$   $h\nu$ -torsion tensor  $C_{\beta\gamma}^\alpha$  will be the Christoffel symbols associated with  $g_x$ . If  $M_x^n$  is the tangent vector space to  $M^n$  at  $x^i$

(=  $u^\alpha$ ) the  $(M_x^{n-1}, g_x)$  will be the hypersurface of  $(M_x^n, g_x)$  given by (3.2a) where  $g_x = g_{ij}(x, y) dy^i dy^j$  is the Riemannian metric of  $M_x^n$ . The quantities  $M_{\beta\gamma}$  given in (3.11) will be considered as the coefficients of second fundamental forms of tangent Riemannian space  $(M_x^{n-1}, g_x)$ .

In general the coefficients of the  $r$ th fundamental forms of  $(M^{n-1}, g_x)$  are defined as<sup>13</sup>

$$C_{(1)\alpha\beta} = g_{\alpha\beta}$$

$$C_{(2)\alpha\beta} = M_{\alpha\beta}$$

$$C_{(r)\alpha\beta} = C_{(r-1)\alpha\epsilon} M_{\beta}^{\epsilon} \quad (2 \leq r \leq n)$$

where

$$M_{\beta}^{\epsilon} = g^{\alpha\epsilon} M_{\alpha\beta}.$$

#### 4. $h$ -VECTOR FIELDS IN $(M^{n-1}, L)$

At the point of  $(M^{n-1}, L)$ , the vector field  $X_i$  may be written as

$$X_i = X_\alpha B_i^\alpha + \mu N_i \quad \dots(4.1)$$

where

$$(a) \quad X_\alpha = X_i B_\alpha^i \quad (b) \quad \mu = X_i N^i. \quad \dots(4.2)$$

Since  $X_{i|\beta} = X_{i|j} B_\beta^j$ , we have from (4.2 (a)) and (3.10)

$$X_{\alpha|\beta} = X_{i|j} B_\alpha^i B_\beta^j + \mu M_{\alpha\beta}. \quad \dots(4.3)$$

From (3.1) and (3.6) we get

$$L C_{\beta\gamma}^\alpha X_\alpha = L C_{ij}^h X_h B_\beta^i B_\gamma^j - L \mu M_{\beta\gamma}. \quad \dots(4.4)$$

If  $X_i$  is a concurrent vector field in  $F_n$  then in view of (2.1), (2.2), (3.2b) and (3.5) eqns. (4.3) and (4.4) reduce to

$$X_{\alpha|\beta} = \mu M_{\alpha\beta}, \quad L C_{\beta\gamma}^\alpha X_\alpha = \rho h_{\beta\gamma} - L \mu M_{\beta\gamma}$$

These relations yield the



*Theorem 4.1*—If  $X_i$  is an  $h$ -vector field in  $(M^n, L)$  the vector field  $X_\alpha = X_i B_\alpha^i$  is also an  $h$ -vector field in  $(M^{n-1}, L)$  if and only if

(i)  $X_i$  is tangential to the hypersurface  $(M^{n-1}, L)$

or

(ii)  $M_{\alpha\beta} = 0$ .

The hyperplane of first, second and third kinds are defined<sup>6</sup>. In a hyper plane of third kind  $M_{\alpha\beta}$  vanishes<sup>6</sup>. Thus :

*Theorem 4.2*—If  $X_i$  is an  $h$ -vector field in  $(M^n, L)$  then vector field  $X_i B_\alpha^i$  is also an  $h$ -vector field in a hyperplane of third kind.

In the following we assume that  $X_i$  is tangential to  $(M^{n-1}, L)$ , so that

$$(a) \quad X_i = X_\alpha B_i^\alpha \quad (b) \quad X^i = X^\alpha B_\alpha^i \quad \dots(4.5)$$

where

$$X^\alpha = g^{\alpha\beta} X_\beta.$$

## 5. THE $n$ -FUNDAMENTAL FORMS OF HYPERSURFACE OF $(M^n, L^*)$

Let  $(M^{n-1}, L^*)$  be a hypersurface of  $(M^n, L^*)$  given by the same equation (3.1). The relations (2.8), (2.9), (3.2b), (3.3) and (4.5) yield

$$g_{\alpha\beta}^* = \sigma g_{\alpha\beta} (1 - \sigma) l_\alpha l_\beta + X_\alpha X_\beta \quad \dots(5.1)$$

$$g^{*\alpha\beta} = \sigma^{-1} g^{\alpha\beta} - \frac{(1 - \sigma)}{L\lambda} (l^\alpha X^\beta + l^\beta X^\alpha) \\ + (1 - \sigma) \frac{(X^2 + \sigma^2)}{\lambda} l^\alpha l^\beta + \frac{1}{\lambda} X^\alpha X^\beta. \quad \dots(5.2)$$

From (3.2b), (3.3) and (2.5) we also have

$$h_{\alpha\beta} = h_{ij} B_\alpha^i B_\beta^j, \quad h_{ij} B_\alpha^i N^j = 0 \quad \dots(5.3)$$

and

$$m_{\alpha\beta} = X_\alpha - \beta/L l_\alpha = m_i B_\alpha^i. \quad \dots(5.4)$$

It is to be noted that if  $N^{*i}$  is a unit normal vector to  $(M^{n-1}, L^*)$  then it is not normal to  $(M^{n-1}, L)$ .

We may write

$$N^{*i} = N^\alpha B_\alpha^i + D N^i. \quad \dots(5.5)$$

To obtain  $N^\alpha$  and  $D$  we use (3.8), (3.5) and (5.1). Thus we get

$$g^{*\alpha\beta} N^\alpha + D_\mu X_\beta = 0 \quad \dots(5.6)$$

$$g^{*\alpha\beta} N^\alpha N^\beta + D^2 \sigma + \mu (2D X_\alpha N^\alpha + D^2 \mu) = 1. \quad \dots(5.7)$$

If  $X_i$  is tangential to  $(M^{n-1}, L)$  then  $\mu = 0$  and eqns. (5.6) and (5.7) give  $N^\alpha = 0$ ,  $D = \sigma^{-1/2}$

From (5.5) it follows that

$$N^{*i} = \sigma^{-1/2} N^i. \quad \dots(5.8)$$

Hence we have the following :

*Theorem 5.1*—Let  $(M^n, L^*)$  be a Finsler space obtained from  $(M^n, L)$  by the transformation (2.4). If  $(M^{d-1}, L^*)$  and  $(M^{n-1}, L)$  are hypersurfaces of these spaces and  $X_i$  is tangential to the hypersurface  $(M^{n-1}, L)$  then the vector normal to  $(M^{n-1}, L)$  is also normal to  $(M^{n-1}, L^*)$ .

Now we establish the following :

*Theorem 5.2*—Let  $(M^n, L^*)$  be a locally Minkowskian space obtained from locally Minkowskian space  $(M^n, L)$  by the transformation (2.4). Let  $(M^{n-1}, L^*)$  and  $(M^{n-1}, L)$  be hyper surfaces of  $(M^n, L^*)$  and  $(M^d, L)$  respectively. If  $X_i$  is tangential to the hypersurfaces  $(M^{n-1}, L)$  and  $(M_x^{n-1}, g_x)$ ,  $(M^n, g_x^*)$ ,  $(M_x^{n-1}, g_x)$ ,  $(M_x^{n-1}, g_x^*)$  are tangent Riemannian spaces to  $(M^n, L)$ ;  $(M^n, L^*)$ ,  $(M^{n-1}, L)$ ,  $(M^{n-1}, L^*)$  respectively, then we have the following :

(i) Second fundamental forms of  $(M_x^{n-1}, g_x)$  and  $(M_x^{n-1}, g_x^*)$  are proportional

(ii) Every asymptotic direction of  $(M_x^{n-1}, g_x)$  is asymptotic direction of  $(M_x^{n-1}, g_x^*)$ .

(iii) The  $r$ th fundamental tensors of  $(M_x^{n-1}, g_x)$  and  $(M_x^{n-1}, g_x^*)$  are related by

$$C^{*(r)}_{\alpha\beta} = \sigma^{3-r/2} [C^{(r)}_{\alpha\beta} + \sum_{m=2}^{r-1} P^{(m)}_{\beta} Q_{(r+m-1)\alpha}], \quad 3 \leq r \leq n \quad \dots(5.9)$$

where

$$P^{(m)}_{\alpha} = \sqrt{\frac{\sigma}{\lambda}} C^{(m)}_{\alpha\epsilon} X^\epsilon, \quad 2 \leq m \leq n-1 \quad \dots(5.10)$$



$$R_{(m)} = \sqrt{\frac{\sigma}{\lambda}} P_{(m)\epsilon} X^\epsilon, \quad 2 \leq m \leq n-1 \quad \dots(5.11)$$

$$Q_{(2)\alpha} = P_{(2)\alpha} \quad \dots(5.12a)$$

$$Q_{(r)\alpha} = P_{(r)\alpha} + \sum_{m=2}^{r-1} R_{(m)} Q_{(r+1-m)\alpha}, \quad 3 \leq r \leq n-1 \quad \dots(5.12b)$$

PROOF : (i) If  $X_t$  is tangential to the hypersurface to  $(M^{n-1}, L)$  then  $\mu = 0$  and hence

$$m_t N^t = 0.$$

Thus from (2.13), (3.10), (5.3), (5.7) it follows that

$$M_{\beta\gamma} = \sigma^{1/2} M_{\beta\gamma}. \quad \dots(5.13)$$

This proves (i).

(ii) A direction  $t^\alpha$  for which  $M_{\alpha\beta} t^\alpha t^\beta = 0$  is said to be an asymptotic direction. In view of this definition and (5.13) we get (ii).

(iii) The validity of relation (5.9) is established by induction.

Since  $C_{ijk}$  is an indicatory tensor from (3.2b) and (3.11) it follows that  $M_{\beta\gamma} l^\gamma = 0$ . Hence from (3.12) we get

$$C_{(r)\beta\gamma} l^\gamma = 0 = C_{(r)\beta\gamma} l^\beta, \quad 2 \leq \gamma \leq n. \quad \dots(5.14)$$

Hence from (5.10), (5.11), (5.12) we get

$$P_{(r)\alpha} l^\alpha = 0, \quad Q_{(r)\alpha} l^\alpha = 0, \quad 2 \leq r \leq n. \quad \dots(5.15)$$

From (5.2), (5.13) and (5.4), we get

$$M_{\beta}^{*\alpha} = \sigma^{-1/2} \left[ M_{\beta}^{\alpha} - \frac{(1-\sigma)\beta\sigma l^{\alpha}}{L\lambda} - \frac{\sigma}{\lambda} X^{\alpha} \right] M_{\beta\gamma} X^{\gamma}. \quad \dots(5.16)$$

The relations (3.12), (5.10), (5.14), (5.15) and (5.16) yield

$$C_{(3)\alpha\beta}^{*} = C_{(3)\alpha\beta} + \sqrt{\frac{\sigma}{\lambda}} P_{(2)\alpha} P_{(2)\beta}. \quad \dots(5.17)$$

From (5.12a) and (5.17), it is evident that (5.9) holds for  $r = 3$ . For a given fixed value of the integer  $s$  with  $3 \leq s \leq n-1$ , we have

$$C_{(s+1)\alpha\beta}^{*} = C_{(s)\alpha\beta}^{*} M_{\beta}^{*\epsilon}. \quad \dots(5.18)$$

Now let us suppose that (5.9) is valid for  $s = 3, 4, 5, \dots, r$ , so that we can write (5.18) in the form

$$C_{(s+1)\alpha\beta}^* = \sigma^{(3-s)/2} \left[ C_{(s)\alpha\beta} + \sum_{m=2}^{s-1} P_{(m)\beta} Q_{(s+1-m)} \right] M_{\beta}^{*s}$$

which in view of (5.10), (5.11), (5.12a), (5.14), (5.15) and (5.16) gives

$$\begin{aligned} C_{(s+1)\alpha\beta}^* &= \sigma^{(2-s)/2} \left[ C_{(s-1)\alpha\beta} + \sum_{m=2}^{s-1} P_{(m+1)\beta} Q_{(s+1-m)\alpha} \right. \\ &\quad \left. + P_{(2)\beta} \{P_{(2)s} + \sum_{m=2}^{s-1} R_{(m)} Q_{(s+1-m)\alpha}\} \right] \\ &= \sigma^{(2-s)/2} [C_{(s+2)\alpha\beta} + \sum_{m=2}^s P_{(m)\beta} Q_{(s+2-m)\alpha}]. \end{aligned}$$

This show that (5.9) is valid for  $r = s + 1$ , which completes is proof of (iii).

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## REMARKS ON SUBMANIFOLDS OF CODIMENSION 2 OF AN EVEN-DIMENSIONAL EUCLIDEAN SPACE

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We determine submanifolds  $M$  of codimension 2 in an even-dimensional Euclidean space  $E$  in relation to the integrability of the almost complex structure  $J$  on  $M \times R^1 \times R^1$ .

### INTRODUCTION

Blair *et al.*<sup>1</sup> introduced the so-called  $(f, g, u, v, \lambda)$ -structure which is naturally defined in a submanifold of codimension 2 of an almost complex manifold. An even-dimensional sphere of codimension 2 of an even-dimensional Euclidean space is a typical example of a manifold admitting the structure. Okumura<sup>2</sup>, Yano and Okumura<sup>4</sup> studied submanifolds admitting a normal  $(f, g, u, v, \lambda)$ -structure.

In the present paper, we shall study submanifolds of codimension 2 in an even-dimensional Euclidean space in relation to the Nijenhuis tensors formed by the induced  $(f, g, u, v, \lambda)$ -structure on the submanifolds.

### 1. SUBMANIFOLDS OF CODIMENSION 2 OF AN EVEN-DIMENSIONAL EUCLIDEAN SPACE

Let  $E$  be a  $(2n + 2)$ -dimensional Euclidean space and  $X$  the position vector of a point of  $E$  from the origin of  $E$ . Since  $E$  is even-dimensional, it is regarded as a flat Kaehlerian manifold, that is, there exists a  $(1, 1)$ -tensor field  $F$  satisfying

$$F^2 = -I, \quad FY \cdot FZ = Y \cdot Z \quad \dots(1.1)$$

for any vectors  $Y$  and  $Z$  and

$$\tilde{\nabla} F = 0 \quad \dots(1.2)$$

where  $I$  denotes the identity transformation, a dot the inner product in  $E$  and  $\tilde{\nabla}$  the Riemannian connection of  $E$ .

Let  $M$  be a  $2n$ -dimensional submanifold of  $E$  covered by a system of local coordinate neighbourhoods  $\{U; x^h\}$ , where and in the sequel the indices  $h, i, j, k, \dots$  run over the range  $\{1, 2, \dots, 2n\}$ .

By putting  $X_i = \partial_i X$  ( $\partial_i = \partial/\partial x^i$ ),  $X_i$  are  $2n$  linearly independent local vector fields tangent to  $M$  and the induced Riemannian metric of  $M$  is given by  $g_{ji} = X_j, X_i$ .

We denote by  $C$  and  $D$  two mutually orthogonal unit normals to  $M$  such that  $X_i, C$  and  $D$  form the positive orientation of  $E$ .

The transforms  $FX_i, FC$  and  $FD$  of  $X_i, C$  and  $D$  by  $F$  can be expressed as

$$FX_i = f_i^h X_h + u_i C + v_i D \quad \dots(1.3)$$

$$FC = -u^h X_h + \lambda D$$

$$FD = -v^h X_h - \lambda C \quad \dots(1.4)$$

where  $f_i^h$  are components of a tensor field of type  $(1, 1)$ ,  $u_i$  and  $v_i$  1-forms,  $\lambda$  a function of  $M$ ,  $u^h = u_i g^{ih}$  and  $v^h = v_i g^{ih}$ .

Applying  $F$  to (1.3), (1.4) and (1.5) and taking account of (1.1) and (1.3)~(1.5) we have<sup>2,4</sup>.

$$f_i^t f_t^h = -\delta_i^h + u_i u^h + v_i v^h \quad \dots(1.6)$$

$$u_i f_i^t = \lambda v_t, v_t f_t^i = -\lambda u_i \quad \dots(1.7)$$

$$u_i u^i = 1 - \lambda^2 = v_i v^i, u_i v^i = 0 \quad \dots(1.8)$$

$$f_j^k f_i^h g_{kh} = g_{ji} - u_j u_i - v_j v_i \quad \dots(1.9)$$

and  $f_{ji} = f_j^h g_{hi}$  is skew symmetric in  $i$  and  $j$

The totality  $(f, g, u, v, \lambda)$  satisfying eqns. (1.6) ~ (1.9) is called an  $(f, g, u, v, \lambda)$ -structure<sup>1</sup>.

The equations of Gauss and Weingarten are given by

$$\nabla_j X_i = h_{ji} C + k_{ji} D \quad \dots(1.10)$$

$$\nabla_j C = -h_j^h X_h + l_j D \quad \dots(1.11)$$

$$\nabla_j D = -k_j^h X_h - l_j C \quad \dots(1.12)$$

where  $h_{ji}$  and  $k_{ji}$  are the second fundamental tensors of  $M$  with respect to the normals  $C$  and  $D$  respectively, and  $l_j$  the third fundamental tensor of  $M$  in  $E$  and  $\nabla$  the Riemannian connection of  $M$ .



Differentiating eqns. (1.3) ~ (1.5) and taking account of (1.2), (1.10), (1.11) and (1.12), we have<sup>2,4</sup>,

$$\nabla_j f_i^h = -h_{ji} u^h + h_j^h u_i - k_{ji} v^h + k_j^h v_i \quad \dots(1.13)$$

$$\nabla_j u_i = -h_{jk} f_i^k - \lambda k_{ji} + l_j v_i \quad \dots(1.14)$$

$$\nabla_j v_i = -k_{jk} f_i^k + \lambda h_{ji} - l_j u_i \quad \dots(1.15)$$

$$\nabla_j \lambda = -h_j^k v_k + k_j^k u_k. \quad \dots(1.16)$$

## 2. NIJENHUIS TENSORS

We consider the product manifold  $M \times R^1 \times R^1$ ,  $R^1$  being a 1-dimensional Euclidean space, and we denote it by  $\bar{M}$ . The indices  $A, B, C, \dots$  will run over the ranges  $1, 2, \dots, 2n, 2n+1, 2n+2$ , and  $*$  stands for  $2n+1$  and  $\#$  for  $2n+2$ . If we define on  $\bar{M}$  a tensor field  $J$  of type (1,1) by local components

$$(J_B^A) = \begin{bmatrix} f_i^j & u_i & v_i \\ -u^j & 0 & \lambda \\ -v^j & -\lambda & 0 \end{bmatrix} \quad \dots(2.1)$$

in each  $\{U \times R^1 \times R^1, x^A\}$ , then we can see that  $J^2 = -I$  on  $\bar{M}$ , that is,  $\{\bar{M}, J\}$  becomes an almost complex manifold. The components of Nijenhuis tensors of the almost complex structure  $J$  are given by

$$N_{ji}^h = f_j^k \partial_k f_i^h - f_i^k \partial_k f_j^h - (\partial_j f_i^k - \partial_i f_j^k) f_k^h \quad \dots(2.2)$$

$$+ (\partial_j u_i - \partial_i u_j) u^h + (\partial_j v_i - \partial_i v_j) v^h$$

$$N_{ji}^* = f_j^k \partial_k u_i - f_i^k \partial_k u_j - (\partial_j f_i^k - \partial_i f_j^k) u_k \quad \dots(2.3)$$

$$+ \lambda (\partial_j v_i - \partial_i v_j)$$

$$N_{ji}^\# = f_j^k \partial_k v_i - f_i^k \partial_k v_j - (\partial_j f_i^k - \partial_i f_j^k) v_k \quad \dots(2.4)$$

$$- \lambda (\partial_j u_i - \partial_i u_j)$$

$$N_{j*}^h = -f_j^k \partial_k v^h - v^k \partial_k f_j^h + (\partial_j u^k) f_k^h + (\partial_i \lambda) v^h \quad \dots(2.5)$$

$$N_{j\#}^h = -f_j^k \partial_k v^h + v^k \partial_k f_j^h + (\partial_j v^k) f_k^h - (\partial_j \lambda) v^h$$

$$N_{\#*}^h = v^k \partial_k u^h - u^k \partial_k v^h \quad \dots(2.7)$$

$$N_{*i}^{\#} = -u^k \partial_k v_i - f_i^k \partial_k \lambda - (\partial_i u^k) v_k \quad \dots(2.8)$$

$$N_{*i}^* = -u^k \partial_k u_i - (\partial_i u^k) u_k - \lambda (\partial_i \lambda) \quad \dots(2.9)$$

$$N_{j\#}^* = -f_j^k (\partial_k \lambda) - v^k (\partial_k u_j) + (\partial_j v^k) u_k \quad \dots(2.10)$$

$$N_{j\#}^{\#} = v^k (\partial_k v_j) + (\partial_j v^k) v_k + \lambda (\partial_j \lambda) \quad \dots(2.11)$$

$$N_{\#*}^* = -u^k (\partial_k \lambda) \quad \dots(2.12)$$

$$N_{\#*}^{\#} = -v^k (\partial_k \lambda). \quad \dots(2.13)$$

The Nijenhuis tensor of an almost complex structure  $J$  satisfies the identity<sup>3</sup>

$$N_{CE}^A J_B^E + N_{CB}^E J_E^A = 0.$$

Hence, substituting (2.1) into (2.4) and taking account of the algebraic properties of  $(f, g, u, v, \lambda)$ -structure, we can obtain the

*Proposition 2.1*—Let there be given an  $(f, g, u, v, \lambda)$  structure on  $M$  and let the function  $\lambda(1 - \lambda^2)$  does not vanish almost everywhere. If  $N_{ji}^h$ ,  $N_{ji}^*$  and  $N_{ji}^{\#}$  vanish on  $M$ , so do the other components of the Nijenhuis tensors of  $J$ .

### 3. SUBMANIFOLDS WITH VANISHING NIJENHUIS TENSORS

Let  $M$  be a submanifold of codimension 2 in a  $(2n + 2)$ -dimensional Euclidean space  $E$ . Then  $M$  admits an  $(f, g, u, v, \lambda)$ -structure and the equations of Gauss, Codazzi and Ricci on  $M$  are given by

$$K_{j\#}^h = h_k^h h_{ji} - h_j^h h_{ki} + k_k^h k_{ji} - k_j^h k_{ki} \quad \dots(3.1)$$

$$\nabla_k h_{ji} - \nabla_j h_{ki} - l_k k_{ji} + l_j k_{ki} = 0 \quad \dots(3.2)$$

$$\nabla_k k_{ji} - \nabla_j k_{ki} + l_k h_{ji} - l_j h_{ki} = 0 \quad \dots(3.3)$$

$$\nabla_j l_i - \nabla_i l_j + h_j^t k_{ti} - h_j^t k_{it} - h_i^t k_{tj} = 0 \quad \dots(3.4)$$

respectively.

Assume that the function  $\lambda(1 - \lambda^2)$  does not vanish almost everywhere on  $M$ . In the case where the connection induced in the bundle of  $M$  has null curvature, we say that the normal connection is trivial.

The following lemma is well known<sup>4</sup>.



**Lemma 3.1**—If the tensor  $N_{jt}^h$  vanishes on  $M$  and the connection induced in the bundle is trivial, then there exist two constants  $\alpha$  and  $\beta$  such that

$$h_j^t u^j = \alpha u^t, h_j^t v^j = \alpha v^t \quad \dots(3.5)$$

$$k_j^t u^j = \beta u^t, k_j^t v^j = \beta v^t \quad \dots(3.6)$$

$$h_j^h h_{th} = \alpha h_{jt}, k_j^h k_{th} = \beta k_{jt} \quad \dots(3.7)$$

and  $h_i^h$  and  $k_i^h$  commute with  $f_i^h$ .

On the other hand, the tensor fields (2.2)  $\sim$  (2.4) can be rewritten as

$$\begin{aligned} N_{jt}^h & (f_j^t h_t^h - h_j^t f_t^h) u_t - (f_i^t h_t^h h_t^h - h_i^t f_t^h) u_j \\ & + (f_j^t k_t^h - k_j^t f_t^h) v_t - (f_i^t k_t^h - k_i^t f_t^h) v_j \\ & + (l_j v_t - l_t v_j) u^h - (l_j u_t - l_t u_j) v^h. \end{aligned} \quad \dots(3.8)$$

$$\begin{aligned} N_{jt}^* & = l_k (f_j^k v_t - f_i^k v_j) + \lambda (l_t u_t - l_j u_t) \\ & - (h_j^k u_t + k_j^k v_t - h_i^k u_j - k_i^k v_j) u_k \end{aligned} \quad (3.9)$$

$$\begin{aligned} N_{ji}^{\#} & = l_k (f_i^k u_j - f_j^k u_t) - \lambda (l_j v_t - l_t v_j) \\ & - v_k (h_j^k u_t + k_j^k v_t - h_i^k u_j - k_i^k v_j). \end{aligned} \quad \dots(3.10)$$

If  $N_{jt}^* = 0$  in addition to the assumptions of Lemma 3.1, then we have

$$\beta u_j v_t - \beta u_t v_j = 0. \quad \dots(3.11)$$

Similarly,  $N_{ji}^{\#} = 0$  implies

$$\alpha v_j u_t - \alpha v_t u_j = 0. \quad \dots(3.12)$$

**Lemma 3.2**—If the tensor  $N_{jt}^h$  vanishes on  $M$  and the connection induced in the normal bundle is trivial, then the following condition are equivalent to one another :

- (1) The components  $N_{jt}^*$  vanishes.
- (2)  $\beta = 0$ .
- (3)  $k_{jt} = 0$ .

In this case, the equations of Gauss and Weingarten reduce to

$$\nabla_j X_i = h_{ji} C, \quad \nabla_j C = -h_j^h X_h, \quad \nabla_j D = 0$$

respectively, and consequently  $D$  is a constant vector and we have

$$\nabla_j (X \cdot D) = 0$$

that is,  $X \cdot D = \text{constant}$ . Thus we have

*Theorem 3.3*—Let  $M$  be a submanifold of codimension 2 of  $E^{2n+2}$  and suppose the connection induced in the normal bundle of  $M$  is trivial. If  $N_{ji}^h$  and  $N_{ji}^*$  vanish, then  $M$  is a hypersurface of a hyperplane  $E^{2n+1}$ .

We prepare the following :

*Theorem A<sup>4</sup>*—Let  $M$  be a  $2n$ -dimensional complete differentiable hypersurface in a  $(2n+1)$ -dimensional euclidean space  $E'$ . If the component  $N_{ji}^h$  vanishes on  $M$ , then  $M$  is a product of a sphere and a plane.

Combining Theorems 3.3 and *A*, we obtain :

*Theorem 3.4*—Let  $M$  be a complete submanifold of codimension 2 of  $E^{2n+2}$  such that the connection induced in the normal bundle of  $M$  is trivial. If  $N_{ji}^h$  and  $N_{ji}^*$  vanish, then  $M$  is a product of a sphere and a plane.

Assuming  $N_{ji}^\# = 0$  instead of  $N_{ji}^* = 0$ , we can obtain the same conclusion as Theorem 3.4. Therefore if the components  $N_{ji}^h$ ,  $N_{ji}^*$  and  $N_{ji}^\#$  vanish identically and the connection induced in the normal bundle is trivial, then we get

$$\nabla_j X_i = 0, \quad \nabla_j C = 0, \quad \nabla_j D = 0$$

and

*Theorem 3.5*—Let  $M$  be a submanifold of codimension 2 of  $E^{2n+2}$  with trivial normal connection. If the Nijenhuis tensors  $N_{ji}^h$ ,  $N_{ji}^*$  and  $N_{ji}^\#$  vanish on  $M$ , then  $M$  is a plane of codimension 2.

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## ALMOST IRRESOLUTE FUNCTIONS

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A function  $f: X \rightarrow Y$  is said to be almost irresolute<sup>10</sup> if for each  $x \in X$  and each semi-neighbourhood  $V$  of  $f(x)$ , the semi-closure of  $f^{-1}(V)$  is a semi-neighbourhood of  $x$ . In this paper, we obtain several characterizations of almost irresolute functions and investigate the relationship between such functions and some weak forms of irresolute functions. We also improve on some results established by Dube *et al.*<sup>9</sup>.

### 1. INTRODUCTION

Crossley and Hildebrand<sup>6</sup> defined irresolute functions by utilizing semi-open sets due to Levine<sup>12</sup>. Recently, as weak forms of irresoluteness, weak irresoluteness<sup>9</sup>,  $\theta$ -irresoluteness<sup>9</sup>, almost irresoluteness<sup>21</sup> and quasi irresoluteness<sup>8</sup> have been defined and investigated independently. However, it will turn out that these four weak forms of irresoluteness are equivalent. On the other hand, Dube *et al.*<sup>10</sup> have introduced the notion of almost irresolute functions which is independent of that of almost irresolute functions in the sense of Thakur and Paik<sup>21</sup>.

The purpose of the present paper is to investigate almost irresolute functions in the sense of Dube *et al.*<sup>10</sup>. Note that, after section 3, "almost irresolute" always means "almost irresolute" in the sense of Dube *et al.* In section 2, we point out that weak irresoluteness,  $\theta$ -irresoluteness, almost irresoluteness in the sense of Thakur and Paik and quasi irresoluteness are all equivalent. In section 3, we present several characterizations of almost irresolute functions in the sense of Dube *et al.* In section 4, we investigate the relationship among semi-continuity, quasi irresoluteness and almost irresoluteness in the sense of Dube *et al.*<sup>10</sup>. In section 5, we introduce

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and investigate semi preopen functions. The last section concerns with strongly semi-closed graphs and contains several improvements on results established by Dube *et al.*<sup>9</sup>.

## 2. PRELIMINARIES

In this section, we will point out that almost irresoluteness<sup>21</sup>, weak irresoluteness<sup>9</sup>,  $\theta$ -irresoluteness<sup>9</sup> and quasi irresoluteness<sup>8</sup> are all equivalent.

Throughout the present paper,  $(X, \tau)$  and  $(Y, \sigma)$  (or simply  $X$  and  $Y$ ) denote topological spaces on which no separation axioms are assumed unless explicitly stated. A subset  $S$  of  $X$  is said to be semi-open<sup>12</sup> if there exists an open set  $U$  of  $X$  such that  $U \subset S \subset \text{Cl}(U)$ , or equivalently if  $S \subset \text{Cl}(\text{Int}(S))$ , where  $\text{Cl}(S)$  and  $\text{Int}(S)$  denote the closure of  $S$  and the interior of  $S$ , respectively. A subset  $S$  is called a semi-neighbourhood<sup>3</sup> of a point  $x$  of  $X$  if there exists a semi-open set  $U$  such that  $x \in U \subset S$ . The complement of a semi-open set is called semi-closed. A semi-closed and semi-open set is said to be semi-clopen. For a subset  $S$  of  $X$ , the intersection of all semi-closed sets containing  $S$  is called the semi-closure of  $S$  (Crossley and Hildebrand<sup>5</sup>) and is denoted by  $s\text{Cl}(S)$ . The semi-interior of  $S$ , denoted by  $s\text{Int}(S)$ , is defined by the union of all semi-open sets contained in  $S$ . The family of all semi-open sets of  $X$  is denoted by  $SO(X)$ . For each  $x \in X$ , the family of all semi-open sets containing  $x$  is denoted by  $SO(X, x)$ . A subset  $S$  is said to be regular-open if  $S = \text{Int}(\text{Cl}(S))$ . A subset  $S$  is said to be regular semi-open<sup>4</sup> if there exists a regular open set  $U$  of  $X$  such that  $U \subset S \subset \text{Cl}(U)$ .

The following two lemmas are due to Di Maio and Noiri<sup>7</sup>.

**Lemma 2.1**—The following are equivalent for a subset  $A$  of a space  $X$ :

- (a)  $A$  is regular semi-open.
- (b)  $A = s\text{Int}(s\text{Cl}(A))$ .
- (c)  $A$  is semi-clopen.

**Lemma 2.2**—If  $A \in SO(X)$ , then  $s\text{Cl}(A)$  is semi-clopen.

**Definition 2.3**—A function  $f: (X, \tau) \rightarrow (Y, \sigma)$  is said to be irresolute<sup>6</sup> (resp. semi-continuous<sup>12</sup>) if  $f^{-1}(V) \in SO(X, \tau)$  for every  $V \in SO(Y, \sigma)$  (resp.  $V \in \sigma$ ).

**Definition 2.4**—A function  $f: X \rightarrow Y$  is said to be (a) almost irresolute<sup>21</sup> if  $f^{-1}(V) \in SO(X)$  for every regular semi-open set  $V$  of  $Y$ ; (b) weakly irresolute<sup>9</sup> (resp.  $\theta$ -irresolute<sup>9</sup>) if for each  $x \in X$  and each semi-neighbourhood  $V$  of  $f(x)$ , there exists a semi-neighbourhood  $U$  of  $x$  such that  $f(U) \subset s\text{Cl}(V)$  (resp.  $f(s\text{Cl}(U)) \subset s\text{Cl}(V)$ ); (c) quasi irresolute<sup>8</sup> if for  $x \in X$  and each  $V \in SO(Y, f(x))$ , there exists  $U \in SO(X, x)$  such that  $f(U) \subset s\text{Cl}(V)$ .

**Lemma 2.5**<sup>8</sup>—The following are equivalent for a function  $f: X \rightarrow Y$ :



- (a)  $f$  is quasi irresolute.
- (b) For each  $x \in X$  and each  $V \in SO(Y, f(x))$ , there exists  $U \in SO(X, x)$  such that  $f(s\text{Cl}(U)) \subset s\text{Cl}(V)$ .
- (c)  $f^{-1}(V)$  is semi-clopen in  $X$  for every semi-clopen set  $V$  of  $Y$ .
- (d)  $f^{-1}(V) \subset s\text{Int}(f^{-1}(s\text{Cl}(V)))$  for every  $V \in SO(Y)$ .
- (e)  $s\text{Cl}(f^{-1}(V)) \subset f^{-1}(s\text{Cl}(V))$  for every  $V \in SO(Y)$ .

*Theorem 2.6*— The following are equivalent for a function  $f: X \rightarrow Y$ :

- (a)  $f$  is quasi irresolute.
- (b)  $f$  is weakly irresolute.
- (c)  $f$  is  $\theta$ -irresolute.
- (d)  $f$  is almost irresolute.

*PROOF*: This follows from Definition 2.4, Lemmas 2.1 and 2.5.

*Remark 2.7*: It is shown in Di Maio and Noiri<sup>8</sup> that semi-continuity and quasi irresoluteness are independent of each other and they are implied by irresoluteness.

### 3. CHARACTERIZATIONS

In this section, we obtain several characterizations of almost irresolute functions in the sense of Dube *et al.*<sup>10</sup>. A subset  $S$  of a space  $X$  is said to be preopen<sup>15</sup> if  $S \subset \text{Int}(\text{Cl}(S))$ .

*Definition 3.1*— A subset  $S$  of a space  $X$  is said to be semi-preopen<sup>1</sup> if there exists a preopen set  $U$  in  $X$  such that  $U \subset S \subset \text{Cl}(U)$ .

The family of all semi-preopen sets in  $X$  is denoted by  $SPO(X)$ . The complement of a semi-preopen set is called semi-preclosed<sup>1</sup>.

*Lemma 3.2*<sup>1</sup>—The following are equivalent for a subset  $A$  of a space  $X$ .

- (a)  $A \in SPO(X)$ .
- (b)  $A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$ .
- (c)  $A \subset s\text{Int}(s\text{Cl}(A))$ .

*PROOF*: This follows from Theorems 2.4 and 3.21 of Andrijević<sup>1</sup>.

*Definition 3.3*— A function  $f: X \rightarrow Y$  is said to be almost irresolute<sup>10</sup> if for each  $x \in X$  and each semi-neighbourhood  $V$  of  $f(x)$ ,  $s\text{Cl}(f^{-1}(V))$  is a semi-neighbourhood of  $x$ .

Henceforth, “almost irresolute” always means “almost irresolute” in the sense of Definition 3.3, that is, Dube *et al.*<sup>10</sup>.

**Theorem 3.4**— The following are equivalent for a function  $f: X \rightarrow Y$ :

- (a)  $f$  is almost irresolute.
- (b)  $f^{-1}(V) \subset s \text{ Int } (s \text{ Cl } (f^{-1}(V)))$  for every  $V \in SO(Y)$ .
- (c)  $f^{-1}(V) \subset \text{Cl } (\text{Int } (\text{Cl } (f^{-1}(V))))$  for every  $V \in SO(Y)$ .
- (d)  $f^{-1}(V) \in SPO(X)$  for every  $V \in SO(Y)$ .

**PROOF:** (a)  $\Rightarrow$  (b): Let  $V \in SO(Y)$  and  $x \in f^{-1}(V)$ . Since  $V$  is a semi-neighbourhood of  $f(x)$ ,  $s \text{ Cl } (f^{-1}(V))$  is a semi-neighbourhood of  $x$  and hence there exists  $U \in SO(X, x)$  such that  $U \subset s \text{ Cl } (f^{-1}(V))$ . Therefore, we have  $x \in U \subset s \text{ Int } (s \text{ Cl } (f^{-1}(V)))$ . This implies that  $f^{-1}(V) \subset s \text{ Int } (s \text{ Cl } (f^{-1}(V)))$ .

(b)  $\Rightarrow$  (a): Let  $x \in X$  and  $V$  be any semi-neighbourhood of  $f(x)$ . There exists  $W \in SO(Y, f(x))$  contained in  $V$ . Therefore, we obtain

$$x \in f^{-1}(W) \subset s \text{ Int } (s \text{ Cl } (f^{-1}(W))) \subset s \text{ Cl } (f^{-1}(W)) \subset s \text{ Cl } (f^{-1}(V)).$$

This implies that  $s \text{ Cl } (f^{-1}(V))$  is a semi-neighbourhood of  $x$ .

It follows from Lemma 3.2 that (b), (c) and (d) are all equivalent.

**Theorem 3.5**— A function  $f: X \rightarrow Y$  is almost irresolute if and only if  $f(s \text{ Cl } (U)) \subset s \text{ Cl } (f(U))$  for every  $U \in SO(X)$ .

**PROOF:** *Necessity*— Let  $U \in SO(X)$ . Suppose that  $y \notin s \text{ Cl } (f(U))$ . There exists  $V \in SO(Y, y)$  such that  $V \cap f(U) = \phi$ ; hence  $f^{-1}(V) \cap U = \phi$ . Since  $U \in SO(X)$ , we have  $s \text{ Int } (s \text{ Cl } (f^{-1}(V))) \cap s \text{ Cl } (U) = \phi$ . By Theorem 3.4,  $f^{-1}(V) \cap s \text{ Cl } (U) = \phi$  and hence  $V \cap f(s \text{ Cl } (U)) = \phi$ . Therefore, we obtain  $y \notin f(s \text{ Cl } (U))$ . This shows that  $f(s \text{ Cl } (U)) \subset s \text{ Cl } (f(U))$ .

*Sufficiency*— Let  $V \in SO(Y)$ . Since  $X - s \text{ Cl } (f^{-1}(V)) \in SO(X)$ , we have  $f(s \text{ Cl } (X - s \text{ Cl } (f^{-1}(V)))) \subset s \text{ Cl } (f(X - s \text{ Cl } (f^{-1}(V))))$  and hence

$$\begin{aligned} X - s \text{ Int } (s \text{ Cl } (f^{-1}(V))) &\subset f^{-1}(s \text{ Cl } (f(X - s \text{ Cl } (f^{-1}(V)))) \\ &\subset f^{-1}(s \text{ Cl } (f(X - f^{-1}(V)))) \subset f^{-1}(s \text{ Cl } (Y - V)) = X - f^{-1}(V). \end{aligned}$$

Therefore, we obtain  $f^{-1}(V) \subset s \text{ Int } (s \text{ Cl } (f^{-1}(V)))$ . It follows from Theorem 3.4 that  $f$  is almost irresolute.

**Theorem 3.6**—The following are equivalent for a function  $f: X \rightarrow Y$ .

- (a)  $f$  is almost irresolute.
- (b) For each  $x \in X$  and each  $V \in SO(Y, f(x))$ , there exists  $U \in SPO(X)$  containing  $x$  such that  $f(U) \subset V$ .
- (c)  $f^{-1}(F)$  is semi-preclosed in  $X$  for every semi-closed set  $F$  of  $Y$ .
- (d)  $\text{Int } (\text{Cl } (\text{Int } (f^{-1}(B)))) \subset f^{-1}(s \text{ Cl } (B))$  for every subset  $B$  of  $Y$ .



(e)  $f(\text{Int}(\text{Cl}(\text{Int}(A)))) \subset s\text{Cl}(f(A))$  for every subset  $A$  of  $X$ .

PROOF : (a)  $\Rightarrow$  (b) : Let  $x \in X$  and  $V \in SO(Y, f(x))$ . Set  $U = f^{-1}(V)$ , then by Theorem 3.4  $U$  is a semi-preopen set containing  $x$  and  $f(U) \subset V$ .

(b)  $\Rightarrow$  (a) : Let  $V \in SO(Y)$  and  $x \in f^{-1}(V)$ . There exists  $U \in SPO(X)$  containing  $x$  such that  $f(U) \subset V$ . By Lemma 3.2, we obtain

$$x \in U \subset s\text{Int}(s\text{Cl}(U)) \subset s\text{Int}(s\text{Cl}(f^{-1}(V)))$$

and hence  $f^{-1}(V) \subset s\text{Int}(s\text{Cl}(f^{-1}(V)))$ . It follows from Theorem 3.4 that  $f$  is almost irresolute.

(a)  $\Rightarrow$  (c) : This is obvious by Theorem 3.4.

(c)  $\Rightarrow$  (d) : Let  $B$  be any subset of  $Y$ . Since  $s\text{Cl}(B)$  is semi-closed,  $f^{-1}(s\text{Cl}(B))$  is semi-preclosed. By utilizing Lemma 3.2, we have

$$\begin{aligned} X - f^{-1}(s\text{Cl}(B)) &\subset \text{Cl}(\text{Int}(\text{Cl}(X - f^{-1}(s\text{Cl}(B))))) \\ &= X - \text{Int}(\text{Cl}(\text{Int}(f^{-1}(s\text{Cl}(B))))) \end{aligned}$$

Therefore, we obtain  $\text{Int}(\text{Cl}(\text{Int}(f^{-1}(B)))) \subset f^{-1}(s\text{Cl}(B))$ .

(d)  $\Rightarrow$  (e) : Let  $A$  be any subset of  $X$ . We have

$$\text{Int}(\text{Cl}(\text{Int}(A))) \subset \text{Int}(\text{Cl}(\text{Int}(f^{-1}(f(A))))) \subset f^{-1}(s\text{Cl}(f(A))).$$

Therefore, we obtain  $f(\text{Int}(\text{Cl}(\text{Int}(A)))) \subset s\text{Cl}(f(A))$ .

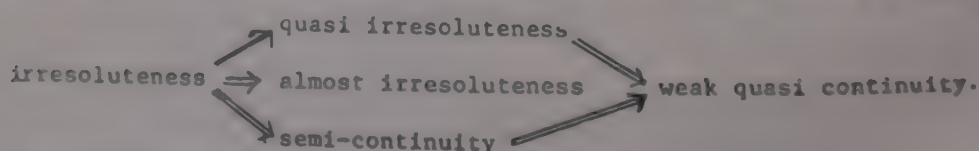
(e)  $\Rightarrow$  (a) : Let  $U \in SO(X)$ . Since  $s\text{Cl}(U) = U \cup \text{Int}(\text{Cl}(U)) = U \cup \text{Int}(\text{Cl}(\text{Int}(U)))$ , we obtain  $f(s\text{Cl}(U)) = f(U) \cup f(\text{Int}(\text{Cl}(\text{Int}(U)))) \subset f(U) \cup s\text{Cl}(f(U)) = s\text{Cl}(f(U))$ . It follows from Theorem 3.5 that  $f$  is almost irresolute.

#### 4. COMPARISONS

In this section, we investigate the relationship among irresoluteness, quasi irresoluteness, almost irresoluteness, semi-continuity and weak quasi continuity.

**Definition 4.1**— A function  $f: X \rightarrow Y$  is said to be weakly quasi continuous<sup>19</sup> if for each  $x \in X$ , each open set  $U$  containing  $x$  and each open set  $V$  containing  $f(x)$ , there exists an open set  $G$  of  $X$  such that  $\emptyset \neq G \subset U$  and  $f(G) \subset \text{Cl}(V)$ .

**Remark 4.2** : The following implications hold for properties on a function :



It is obvious by Lemma 2.2 and Theorem 3.4 that irresoluteness implies almost irresoluteness. The other implications have been shown in section 7 of Di Maio and Noiri<sup>8</sup>.

We shall show that quasi irresoluteness, almost irresoluteness and semi-continuity are respectively independent. The following example shows that almost irresoluteness does not imply weak quasi continuity and hence it implies neither quasi irresoluteness nor semi-continuity.

*Example 4.3—* Let  $X$  be the set of real numbers,  $\tau$  the indiscrete topology for  $X$  and  $\sigma$  the discrete topology for  $X$ . Let  $f: (X, \tau) \rightarrow (X, \sigma)$  be the identity function. Then  $f$  is almost irresolute but it is not weakly quasi continuous.

*Example 4.4—* Let  $X = \{a, b, c\}$  and  $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ . Let  $f: (X, \tau) \rightarrow (X, \tau)$  be a function defined as follows:  $f(a) = f(b) = a$  and  $f(c) = c$ . Then  $f$  is continuous and hence semi-continuous. However,  $f$  is not almost irresolute because there exists  $\{b, c\} \in SO(X, \tau)$  such that  $f^{-1}(\{b, c\}) \notin SPO(X, \tau)$ .

By Examples 4.3 and 4.4, we observe that almost irresoluteness is independent of semi-continuity and also it is independent of weak quasi continuity. The following example and Example 4.3 show that almost irresoluteness and quasi irresoluteness are independent of each other.

*Example 4.5—* Let  $X = \{a, b, c\}$ ,  $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$  and  $\sigma = \{\phi, X, \{c\}\}$ . Let  $f: (X, \tau) \rightarrow (X, \sigma)$  be the identity function. Then  $f$  is quasi irresolute. However, it is not almost irresolute because there exists  $\{c\} \in SO(X, \sigma)$  such that  $f^{-1}(\{c\}) \notin PSO(X, \tau)$ .

It has been shown in Examples 7.2 and 7.3 of Di Maio and Noiri<sup>8</sup> that quasi irresoluteness neither implies semi-continuity nor is implied by semi-continuity.

## 5. SEMI-PREOPEN FUNCTIONS

In this section, we introduce the notion of semi-preopen functions which is independent of both notions of preopen functions and semi-open functions. It will be shown that every quasi irresolute function is almost irresolute if it has one of the following properties: "semi-preopen", "semi-open" and "preopen".

*Definition 5.1—* A function  $f: X \rightarrow Y$  is said to be semi-preopen if  $f(U) \in SPO(Y)$  for every  $U \in SO(X)$ .

*Definition 5.2—* A function  $f: X \rightarrow Y$  is said to be semi-open<sup>2</sup> (resp. preopen<sup>15</sup>) if  $f(U)$  is semi-open (resp. preopen) in  $Y$  for every open set  $U$  of  $X$ .

Rose<sup>20</sup> called preopen functions almost open and showed that  $f: X \rightarrow Y$  is almost open if and only if  $f^{-1}(Cl(V)) \subset Cl(f^{-1}(V))$  for every open set  $V$  of  $Y$ . Therefore, "preopen" is equivalent to "almost open" in the sense of Wilansky<sup>22</sup>.

We shall show that "semi-preopen", "preopen" and "semi-open" are respectively independent. It follows from Examples 1.8 and 1.9 of Noiri<sup>17</sup> that "preopen" and "semi-open" are independent of each other. The following example shows that "open"



does not imply "semi-preopen" in general. Therefore, neither preopeness nor semi-openness implies semi-preopeness.

*Example 5.3—* Let  $X = \{a, b, c, d\}$ ,  $\tau = \{\phi, x, \{a\}, \{a, c\}\}$  and  $\sigma = \{\phi, x, \{a\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\}$ . Let  $f: (X, \tau) \rightarrow (X, \sigma)$  be the identity function. Then  $f$  is an open function. However,  $f$  is not semi-preopen because there exists  $\{a, b\} \in SO(X, \tau)$  such that  $f(\{a, b\}) \notin SPO(X, \sigma)$ .

Let  $f: (X, \tau) \rightarrow (X, \sigma)$  be the function of Example 4.3. Then  $f^{-1}$  is semi-preopen but not semi-open. Moreover, the following example shows that a semi-preopen function is not necessarily preopen.

*Example 5.4—* Let  $X = \{a, b, c\}$ ,  $\tau = \{\phi, X, \{a\}, \{a, b\}\}$  and  $\sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ . Let  $f: (X, \tau) \rightarrow (X, \sigma)$  be a function defined as follows:  $f(a) = a$ ,  $f(b) = c$ , and  $f(c) = b$ . Then  $f$  is semi-preopen. However,  $f$  is not preopen because there exists  $\{a, b\} \in \tau$  such that  $f(\{a, b\})$  is not preopen.

*Theorem 5.5—*The following are equivalent for a function  $f: X \rightarrow Y$ :

- (a)  $f$  is semi-preopen.
- (b)  $f^{-1}(s \text{ Cl}(V)) \subset s \text{ Cl}(f^{-1}(V))$  for every  $V \in SO(Y)$ .
- (c)  $f^{-1}(\text{Int}(\text{Cl}(\text{Int}(B)))) \subset s \text{ Cl}(f^{-1}(B))$  for every subset  $B$  of  $X$ .
- (d)  $f(\text{Int}(A)) \subset \text{Cl}(\text{Int}(\text{Cl}(f(A))))$  for every subset  $A$  of  $X$ .
- (e)  $f(U) \subset \text{Cl}(\text{Int}(\text{Cl}(f(U))))$  for every  $U \in SO(X)$ .
- (f)  $f(U) \subset s \text{ Int}(s \text{ Cl}(f(U)))$  for every  $U \in SO(X)$ .

**PROOF:** (a)  $\Rightarrow$  (b): Let  $V \in SO(Y)$  and  $x \notin s \text{ Cl}(f^{-1}(V))$ . There exists  $U \in SO(X, x)$  such that  $U \cap f^{-1}(V) = \phi$ ; hence  $f(U) \cap V = \phi$ . Therefore, we have  $s \text{ Int}(s \text{ Cl}(f(U))) \cap V = \phi$  and hence  $s \text{ Int}(s \text{ Cl}(f(U))) \cap s \text{ Cl}(V) = \phi$ . Since  $f$  is semi-preopen, by Lemma 3.2 we obtain  $f(U) \cap s \text{ Cl}(V) = \phi$  and  $U \cap f^{-1}(s \text{ Cl}(V)) = \phi$ . Therefore, we have  $x \notin f^{-1}(s \text{ Cl}(V))$  and hence  $f^{-1}(s \text{ Cl}(V)) \subset s \text{ Cl}(f^{-1}(V))$ .

(b)  $\Rightarrow$  (c): Let  $B$  be any subset of  $Y$ . We obtain

$$\begin{aligned} f^{-1}(\text{Int}(\text{Cl}(\text{Int}(B)))) &= f^{-1}(s \text{ Cl}(\text{Int}(B))) \subset s \text{ Cl}(f^{-1}(\text{Int}(B))) \\ &\subset s \text{ Cl}(f^{-1}(B)). \end{aligned}$$

(c)  $\Rightarrow$  (d): Let  $A$  be any subset of  $X$ . Then, we have

$$\begin{aligned} X - f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(f(A)))) &= f^{-1}(\text{Int}(\text{Cl}(\text{Int}(Y - f(A)))) \\ &\subset s \text{ Cl}(f^{-1}(Y - f(A))) \subset s \text{ Cl}(X - A) = X - s \text{ Int}(A). \end{aligned}$$

Therefore, we obtain  $f(s \text{ Int}(A)) \subset \text{Cl}(\text{Int}(\text{Cl}(f(A))))$ .

(d)  $\Rightarrow$  (e): Let  $U \in SO(X)$ . We have  $f(U) = f(s \text{ Int}(U)) \subset \text{Cl}(\text{Int}(\text{Cl}(f(U))))$ .

(e)  $\Rightarrow$  (f) and (f)  $\Rightarrow$  (a): These follow from Lemma 3.2.

**Theorem 5.6**—A quasi irresolute function  $f: X \rightarrow Y$  is almost irresolute if it satisfies one property of the following:

(a) semi-preopen, (b) semi-open, and (c) preopen.

**PROOF:** (a) Suppose that  $f$  is quasi irresolute and semi-preopen. Let  $V \in SO(Y)$ . It follows from Lemma 2.5 that  $f^{-1}(V) \subset s \text{ Int}(f^{-1}(s \text{ Cl}(V)))$ . Moreover, by Theorem 5.5 we obtain  $f^{-1}(V) \subset s \text{ Int}(s \text{ Cl}(f^{-1}(V)))$ . Therefore, it follows from Theorem 3.4 that  $f$  is almost irresolute.

(b) Suppose that  $f$  is quasi irresolute and semi-open. Let  $V \in SO(Y)$ . By Lemma 2.5, we have  $f^{-1}(V) \subset s \text{ Int}(f^{-1}(s \text{ Cl}(V)))$ . It is shown in Theorem 2 of Noiri<sup>16</sup> that  $f$  is semi-open if and only if  $f^{-1}(s \text{ Cl}(B)) \subset \text{Cl}(f^{-1}(B))$  for every subset  $B$  of  $Y$ . Therefore, we obtain  $f^{-1}(V) \subset s \text{ Int}(\text{Cl}(f^{-1}(V))) = \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(V))))$ . By Theorem 3.4,  $f$  is almost irresolute.

(c) Suppose that  $f$  is quasi irresolute and preopen. Let  $V \in SO(Y)$ . By Lemma 2.5, we have  $f^{-1}(V) \subset s \text{ Int}(f^{-1}(s \text{ Cl}(V))) \subset s \text{ Int}(f^{-1}(\text{Cl}(V)))$ . Since  $f$  is preopen, by Theorem 11 of Rose<sup>20</sup> we have

$$f^{-1}(\text{Cl}(V)) = f^{-1}(\text{Cl}(\text{Int}(V))) \subset \text{Cl}(f^{-1}(\text{Int}(V))) \subset \text{Cl}(f^{-1}(V)).$$

Therefore, we obtain  $f^{-1}(V) \subset s \text{ Int}(\text{Cl}(f^{-1}(V))) = \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(V))))$ . By Theorem 3.4,  $f$  is almost irresolute.

**Remark 5.7:** The function  $f$  in Example 4.3 is almost irresolute, open and semi-preopen. However, it is neither quasi irresolute nor semi-continuous.

## 6. STRONGLY SEMI-CLOSED GRAPHS

For a function  $f: X \rightarrow Y$ , the subset  $\{(x, f(x)) \mid x \in X\}$  of the product space  $X \times Y$  is called a graph of  $f$  and is denoted by  $G(f)$ . Dube *et al.*<sup>9</sup> defined and investigated a strongly semi-closed graph. We shall improve on some results established by Dube *et al.*<sup>9</sup>.

**Definition 6.1**—The graph  $G(f)$  is said to be strongly semi-closed Dube<sup>9</sup> if for each,  $(x, y) \notin G(f)$ , there exists  $U \in SO(X, x)$  and  $V \in SO(Y, y)$  such that  $[U \times s \text{ Cl}(V)] \cap G(f) = \emptyset$ .

It follows immediately that  $G(f)$  is strongly semi-closed if and only if for each  $(x, y) \notin G(f)$ , there exist  $U \in SO(X, x)$  and  $V \in SO(Y, y)$  such that  $f(U) \cap s \text{ Cl}(V) = \emptyset$ . The graph  $G(f)$  is said to be semi-closed (resp. closed) if it is semi-closed (resp. closed) in the product space  $X \times Y$ . If  $G(f)$  is strongly semi-closed, then it is semi-closed. However, the converse is false by Example 2 of Dube *et al.*<sup>9</sup>.



A function  $f: X \rightarrow Y$  is said to be pre-semi-open<sup>6</sup> if  $f(U) \in SO(Y)$  for every  $U \in SO(X)$ . Every pre-semi-open function is both semi-open and semi-preopen. However, in Example 4.3,  $f^{-1}$  is semi-preopen but not semi-open and hence not pre-semi-open. Moreover, in Example 5.3,  $f$  is semi-open but not semi-preopen and hence not pre-semi-open.

**Theorem 6.2**— Let  $f: X \rightarrow Y$  be either semi-preopen or semi-open. If  $G(f)$  is closed, then  $G(f)$  is strongly semi-closed.

**PROOF**: Since  $G(f)$  is closed, for each  $(x, y) \notin G(f)$ , there exist open sets  $U$  and  $V$  containing  $x$  and  $y$ , respectively, such that  $f(U) \cap V = \phi$ . First, suppose that  $f$  is semi-preopen. Since  $V$  is open, we obtain  $s \text{Int}(s \text{Cl}(f(U))) \cap s \text{Cl}(V) = \phi$ . Since  $f$  is semi-preopen,  $f(U) \in SPO(Y)$  and hence  $f(U) \cap s \text{Cl}(V) = \phi$  by Lemma 3.2. This shows that  $G(f)$  is strongly semi-closed. Next, suppose that  $f$  is semi-open. Since  $f(U) \in SO(Y)$ , we have  $f(U) \cap s \text{Cl}(V) = \phi$  and hence  $G(f)$  is strongly semi-closed.

**Corollary 6.3**<sup>9</sup>— If  $f: X \rightarrow Y$  is pre-semi-open and  $G(f)$  is closed, then  $G(f)$  is strongly semi-closed.

Noiri<sup>18</sup> showed that a semi-continuous function into a Hausdorff space has a semi-closed graph but it does not have a closed graph in general.

**Theorem 6.4**— If  $f: X \rightarrow Y$  is weakly quasi continuous and  $Y$  is Hausdorff, then  $G(f)$  is strongly semi-closed.

**PROOF**: For each  $(x, y) \notin G(f)$ , there exist disjoint open sets  $V$  and  $W$  of  $Y$  containing  $y$  and  $f(x)$ , respectively. Hence we have  $\text{Cl}(W) \cap \text{Int}(\text{Cl}(V)) = \phi$  and  $\text{Cl}(W) \cap s \text{Cl}(V) = \phi$ . Since  $f$  is weakly quasi continuous, by Theorem 4.1 of Noiri<sup>18</sup> there exists  $U \in SO(X, x)$  such that  $f(U) \subset \text{Cl}(W)$ . Therefore, we obtain  $f(U) \cap s \text{Cl}(V) = \phi$ . This shows that  $G(f)$  is strongly semi-closed.

**Corollary 6.5**<sup>9</sup>— A function  $f: X \rightarrow Y$  has a strongly semi-closed graph if it has one property of the following :

- (a)  $f$  is semi-continuous and  $Y$  is Hausdorff;
- (b)  $f$  is weakly irresolute and  $Y$  is Urysohn;
- (c)  $f$  is  $\theta$ -irresolute and  $Y$  is Urysohn.

**PROOF**: This follows immediately from Theorems 2.6 and 6.4 and Remark 4.2.

A space  $X$  is said to be regular semi- $T_2$  Maheshwari *et al.*<sup>13</sup> (resp. semi- $T_2$  Maheshwari and Prasad<sup>14</sup> if for distinct points  $x, y$  of  $X$ , there exist disjoint regular semi-open (resp. semi-open) sets  $U$  and  $V$  such that  $x \in U$  and  $y \in V$ .

**Lemma 6.6**— A space  $X$  is regular semi- $T_2$  if and only if  $X$  is semi- $T_2$ .

PROOF : Every regular semi- $T_2$  space is obviously semi- $T_2$ . Conversely, suppose that  $X$  is semi- $T_2$ . For distinct points  $x, y$  of  $X$ , there exist  $U \in SO(X, x)$  and  $V \in SO(X, y)$  such that  $U \cap V = \phi$ ; hence  $U \cap sCl(V) = \phi$ . By Lemma 2.2, we have  $sCl(V) \in SO(X)$  and hence  $sCl(U) \cap sCl(V) = \phi$ . It follows from Lemmas 2.1 and 2.2 that  $sCl(U)$  and  $sCl(V)$  are regular semi-open. Therefore,  $X$  is regular semi- $T_2$ .

**Theorem 6.7**— If  $f: X \rightarrow Y$  is quasi irresolute and  $Y$  is semi- $T_2$ , then  $G(f)$  is strongly semi-closed.

PROOF : Let  $(x, y) \notin G(f)$ . Since  $Y$  is semi- $T_2$ , there exist  $V \in SO(Y, y)$  and  $W \in SO(Y, f(x))$  such that  $V \cap W = \phi$ . By Lemma 2.2,  $sCl(V) \cap sCl(W) = \phi$ . Since  $f$  is quasi irresolute, there exists  $U \in SO(X, x)$  such that  $f(U) \subset sCl(W)$ . Therefore, we obtain  $f(U) \cap sCl(V) = \phi$ . This shows that  $G(f)$  is strongly semi-closed.

A surjection  $f: X \rightarrow Y$  is said to be semi- $s$ -connected<sup>9</sup> if  $f^{-1}(V)$  is semi-clopen in  $X$  for every semi-clopen set  $V$  of  $Y$ . It follows from Lemma 2.5 that a surjection is set- $s$ -connected if and only if it is quasi irresolute. Dube *et al.*<sup>9</sup> defined a space  $X$  to be extremally- $s$ -disconnected if  $sCl(U) \in SO(X)$  for every  $U \in SO(X)$ . However, by Lemma 2.2 the semi-closure of a semi-open set is always semi-open.

**Corollary 6.8**<sup>9</sup>— Let  $f: X \rightarrow Y$  be a function and  $Y$  semi- $T_2$ . Then the following hold :

(a) If  $f$  is irresolute, then  $G(f)$  is strongly semi-closed.

(b) If  $f$  is a set- $s$ -connected surjection and  $Y$  is extremally  $s$ -disconnected, then  $G(f)$  is strongly semi-closed.

PROOF : This is an immediate consequence of Remark 4.2 and Theorem 6.7.

**Corollary 6.9**<sup>21</sup>— If  $f: X \rightarrow Y$  is almost irresolute (in the sense of Thakur and Paik) and  $Y$  is regular semi- $T_2$ , then  $G(f)$  is semi-closed.

PROOF : By Theorem 2.6, “quasi irresolute” is equivalent to “almost irresolute” in the sense of Thakur and Paik<sup>21</sup>. Therefore, this result is an immediate consequence of Lemma 6.6 and Theorem 6.7.

Finally, we shall obtain a sufficient condition for a function to be quasi-irresolute. A subset  $S$  of a space  $X$  is said to be  $s$ -closed relative to  $X$ , Di Maio and Noiri<sup>7</sup> if for every cover  $\{V_\alpha \mid \alpha \in \nabla\}$  of  $S$  by semi-open sets of  $X$ , there exists a finite subset  $\nabla_0$  of  $\nabla$  such that  $S \subset \bigcup \{sCl(V_\alpha) \mid \alpha \in \nabla_0\}$ . If  $S = X$ , then the space  $X$  is said to be  $s$ -closed<sup>7</sup>. A space  $X$  is said to be extremally disconnected if the closure of every open set of  $X$  is open in  $X$ .

**Lemma 6.10**— Let  $X$  be extremally disconnected and  $f: X \rightarrow Y$  have a strongly semi-closed graph. If  $K$  is  $s$ -closed relative to  $Y$ , then  $f^{-1}(K)$  is semi-closed in  $X$ .



PROOF: Let  $x \notin f^{-1}(K)$ . For each  $y \in K$ ,  $(x, y) \notin G(f)$  and hence there exist  $U(y) \in SO(X, x)$  and  $V(y) \in SO(Y, y)$  such that  $f(U(y)) \cap sCl(V(y)) = \emptyset$ . Therefore, we have  $U(y) \cap f^{-1}(sCl(V(y))) = \emptyset$  for each  $y \in K$ . Since  $\{V(y) \mid y \in K\}$  is a cover of  $K$  by semi-open sets of  $Y$ , there exists a finite number of points  $y_1, y_2, \dots, y_n$  in  $K$  such that  $K \subset \bigcup \{sCl(V(y_i)) \mid i = 1, 2, \dots, n\}$ . Put  $U = \bigcap \{U(y_i) \mid i = 1, 2, \dots, n\}$ . Since  $X$  is extremally disconnected, it follows from Theorem 2.9 of Janković<sup>11</sup> that  $U \in SO(X, x)$ . Moreover, we have  $U \cap f^{-1}(K) = \emptyset$  and hence  $x \notin sCl(f^{-1}(K))$ . This implies that  $f^{-1}(K)$  is semi-closed in  $X$ .

**Theorem 6.11**— Let  $X$  be extremally disconnected and  $Y$   $s$ -closed. If  $f: X \rightarrow Y$  has a strongly semi-closed graph, then it is quasi irresolute.

PROOF: Let  $V \in SO(Y)$ . By Lemma 2.2,  $sCl(V)$  is semi-clopen. It follows from Propositions 2.3 and 4.2 of Di Maio and Noiri<sup>7</sup> that  $sCl(V)$  is  $s$ -closed relative to  $Y$ . By Lemma 6.10,  $f^{-1}(sCl(V))$  is semi-closed in  $X$ . Therefore, we obtain  $sCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ . It follows from Lemma 2.5 that  $f$  is quasi irresolute.

**Corollary 6.12**— Let  $X$  be extremally disconnected and  $Y$   $s$ -closed semi- $T_2$ . A function  $f: X \rightarrow Y$  is quasi irresolute if and only if  $G(f)$  is strongly semi-closed.

PROOF: This is an immediate consequence of Theorem 6.7 and 6.11.

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## ON SOME DUAL INTEGRAL EQUATIONS INVOLVING BESSEL FUNCTION OF ORDER ONE

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Three different sets of dual integral equations, involving Bessel function of order one, arising in some special axi-symmetric problems of elasticity theory and viscous flow theory are handled for their solutions, either in closed form or in terms of Fredholm integral equations of the second kind. The final results are in full agreement with the ones obtained through the Green's function technique, used previously, for solving the mixed boundary value problems considered here.

### 1. INTRODUCTION

A number of workers have studied mixed boundary value problems, in the axisymmetric case, arising in Elasticity theory associated with punches and cracks<sup>1-3</sup> and in the theory of viscous flows induced by steady rotation or harmonic oscillation of a circular disc<sup>4-8</sup>. Of all possible methods of solution of these axi-symmetric problems, the method of reduction to a set of dual integral equations<sup>1</sup>, appears to be the most natural and straightforward method of attack. But, as pointed out by Shail<sup>3,5</sup> and demonstrated by Stallybrass<sup>9</sup>, certain dual integral equations are not amenable to their solution straightaway. It is for this difficulty that Stallybrass<sup>8</sup> and Shail<sup>3,5,6</sup> have utilised an integral representation of the principal unknown potential by employing a Green's function technique devised specially for the purpose.

We have shown in the present paper that by a suitable use of the Bessel equation itself, it is possible to handle all the problems treated previously via the dual integral equations only and that the details of the Green's function technique can be avoided, if we assume throughout the analysis that  $\int_0^\infty p(s) J_1(sr) ds = \lim_{\epsilon \rightarrow 0} \int_0^\infty e^{-\epsilon s} p(s) J_1(sr) ds$ , wherever such integrals occur. It must be emphasised that the Green's function technique is better-suited to problems associated with more general axi-symmetric bodies than the circular disc for which the dual integral equations are the superior ones.

### 2. THE DUAL INTEGRAL EQUATIONS AND THEIR SOLUTIONS

#### Problem 1

The dual integral equations

$$\int_0^\infty s^2 A(s) J_1(sr) ds = -\sigma(r), \quad 0 < r < a \quad \dots(1)$$



$$\int_0^{\infty} s A(s) J_1(sr) ds = 0, \quad r > a \quad \dots(2)$$

arise (see Erguven<sup>9</sup>) in the study of a static penny-shaped crack problem in a homogeneous isotropic elastic solid under torsion. Here  $\sigma(r)$  represents the distribution of the shear-stress on the face of the crack and it is required that the displacement field given by the integral on the left of eqn. (1b) is zero at  $r = a$ , for the purpose of continuity.

The method of solution of the equations (1) and (2) is as described below, and is different from the method described in Sneddon's book<sup>1</sup>.

We set

$$\int_0^{\infty} s A(s) J_1(sr) ds = f(r), \quad 0 < r < a. \quad \dots(3)$$

Then, using the well-known Hankel's inversion formula to the eqns. (2) and (3), we obtain

$$A(s) = \int_0^a \lambda f(\lambda) J_1(\lambda s) d\lambda. \quad \dots(4)$$

The equations (1) and (4) finally give rise to the following integral equation for the unknown function  $f(\lambda)$ :

$$\int_0^{\infty} s^2 J_1(sr) ds \int_0^a \lambda f(\lambda) J_1(s\lambda) d\lambda = -\sigma(r), \quad (0 < r < a) \quad \dots(5)$$

after interchanging the orders of integration, assuming that such an interchange is permissible here and even later on, for the other two problems treated in this paper.

If we next use the idea that the Bessel function  $J_1(sr)$  satisfies the ordinary differential equation

$$\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} + s^2 \right) J_1(sr) = 0 \quad \dots(6)$$

we observe that under certain special circumstances (as applicable to the crack problem mentioned above), we can rewrite the integral equation (5) in the form

$$\left( r^2 \frac{d^2}{dr^2} + r \frac{d}{dr} - 1 \right) u(r) = r^2 \sigma(r), \quad (0 < r < a) \quad \dots(7)$$

with

$$u(r) = \int_0^{\infty} J_1(sr) ds \int_0^a \lambda f(\lambda) J_1(\lambda s) d\lambda. \quad \dots(8)$$

we then solve the ODE (7) by the standard method of variation of parameters and obtain the general solution in the form

$$u = c_1^0 r + \frac{c_2^0}{r} + \frac{r}{2} \int_0^r \sigma(t) dt - \frac{1}{2r} \int_0^r t^2 \sigma(t) dt \quad \dots(9)$$

where  $c_1^0$  and  $c_2^0$  are arbitrary constants to be determined from the physical considerations of the problem.

Of these two arbitrary constants, the constant  $c_2^0$  must be chosen to be zero in order that  $u$  is finite at  $r = 0$ , and the constant  $c_1^0$  will be settled a little later on.

Next, interchanging the orders of integration in eqn. (9), and using the result (see Shail<sup>5</sup>),

$$\int_0^\infty J_1(sr) J_1(s\lambda) ds = \frac{2}{\pi\lambda r} \int_0^{\min(r,\lambda)} \frac{v^2 dv}{[(r^2 - v^2)(\lambda^2 - v^2)]^{1/2}} \quad \dots(10)$$

We obtain

$$\frac{2}{\pi} \int_0^r \frac{v^2 dv}{(r^2 - v^2)^{1/2}} \int_v^a \frac{f(\lambda) d\lambda}{(\lambda^2 - v^2)^{1/2}} = ru(r), \quad (0 < r < a). \quad \dots(11)$$

Using the Abel's inversion formulae<sup>1</sup>, repeatedly to equation (11), we find that

$$v \int_v^a \frac{f(\lambda) d\lambda}{(\lambda^2 - v^2)^{1/2}} = \int_0^v \frac{d}{dt} (tu(t)) / (v^2 - t^2)^{1/2} dt,$$

and, hence,

$$f(\lambda) = \frac{2\lambda}{\pi a (a^2 - \lambda^2)^{1/2}} \int_0^a \frac{d}{dt} (tu(t)) dt / (a^2 - t^2)^{1/2} - \frac{2\lambda}{\pi} \int_\lambda^a \left( \frac{dn}{dt} \right) dt / (t^2 - \lambda^2)^{1/2} \quad \dots(12)$$

where

$$n(t) = \frac{1}{t} \int_0^a \frac{d}{dm} (mu(m)) / (t^2 - m^2)^{1/2} dm. \quad \dots(13)$$

Equations (12) and (13), along with eqn. (9) completely solve the integral equation (5), if the constant  $c_1^0$  is determined. In order to determine the constant  $c_1^0$ , we need to use the result (12) and the observation that  $f(a) = 0$ , arising out of the physical



requirement involving the continuity of  $f(r)$  at  $r = a$ , as argued earlier. We then obtain the equation that

$$\int_0^a \frac{d}{dt} (tu(t)) dt = 0 \quad \dots(14)$$

and this, together with eqn. (9) serves as the determining equation for the constant  $c_1^0$ .

As a particular case, if we take  $\sigma(r) = cr$ , where  $c$  is a known constant, as considered by Erguven<sup>10</sup>, we easily find that

$$u(r) = c_1^0 + \frac{c}{8} r^2 \quad \dots(15)$$

and eqn. (14) gives that

$$c_1^0 = -\frac{c}{6} a^2 \quad \dots(16)$$

so that eqn. (12) decides that

$$f(\lambda) = \frac{-4c}{3\pi} \lambda (a^2 - \lambda^2)^{1/2} \quad \dots(17)$$

which agrees with the result of Erguven<sup>10</sup>.

### Problem 2

The following dual integral equations arise in the study of the dynamic Reissner-Sagoci problem considered by Shail<sup>3</sup>:

$$\int_0^\infty c(\xi) J_1(\xi\rho) d\xi = \frac{\alpha\rho}{p}, \quad (0 \leq \rho < 1) \quad \dots(18)$$

$$\int_0^\infty (\xi^2 + p^2/\beta^2)^{1/2} C(\xi) J_1(\xi\rho) d\xi = 0, \quad (\rho > 1). \quad \dots(19)$$

In order to derive the above dual equations, we have used the representation of the solution of the mixed boundary value problem of Shail<sup>3</sup>, in the form:

$$\bar{v}(\rho, z) = \int_0^\infty C(\xi) e^{-(\rho^2/\beta^2 + \xi^2)^{1/2} z} J_1(\xi\rho) d\xi, \quad \begin{cases} \rho \geq 0 \\ z > 0 \end{cases} \quad \dots(20)$$

which must satisfy the p.d.e. and the boundary conditions as given by

$$\frac{\partial^2 \bar{v}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \bar{v}}{\partial \rho} - \frac{\bar{v}}{\rho^2} + \frac{\partial^2 \bar{v}}{\partial z^2} - \frac{p^2 \bar{v}}{\beta^2} = 0, \quad z > 0 \quad \dots(21)$$

and

$$\left. \begin{aligned} \bar{v} &= \frac{\alpha \rho}{\rho}, (0 \leq \rho < 1) \\ \frac{\partial \bar{v}}{\partial t} &= 0, (\rho > 1) \end{aligned} \right\} \text{on } z = 0. \quad \dots(22)$$

Shail<sup>3</sup> has used, like Stallybrass<sup>9</sup>, a special Green's function technique to solve the mixed problems (21) and (22), as no direct method of attack exists to solve the dual eqns. (18) and (19). We present an approach, which is similar to the one employed for Problem 1 above, utilizing the Bessel equation (6), and show that for large values of  $\rho$ , the solution of the dual equations (18) and (19) can be determined exactly in the same manner as described by Shail<sup>3</sup>.

We set

$$C(\xi) = \left( \frac{\rho^2}{\beta^2} + \xi^2 \right)^{1/2} D(\xi) \quad \dots(23)$$

and rewrite eqns. (18) and (19) in the form

$$\int_0^\infty \left( \frac{\rho^2}{\beta^2} + \xi^2 \right)^{1/2} D(\xi) J_1(\xi \rho) d\xi = \frac{\alpha \rho}{\rho}, (0 \leq \rho < 1) \quad \dots(23)$$

and

$$\int_0^\infty \left( \frac{\rho^2}{\beta^2} + \xi^2 \right) D(\xi) J_1(\xi \rho) d\xi = 0 (\rho > 1). \quad \dots(25)$$

Equation (25) can be recast in the form

$$\left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{1}{\rho^2} - \frac{\rho^2}{\beta^2} \right) \int_0^\infty D(\xi) J_1(\xi \rho) d\xi = 0, (\rho > 1) \quad \dots(26)$$

after using the Bessel equation (6), and this, on integration gives that

$$\int_0^\infty D(\xi) J_1(\xi \rho) d\xi = C_0 K_1(\rho \rho / \beta), (\rho > 1) \quad \dots(27)$$

after neglecting the part of the solution involving  $I_1(\rho \rho / \beta)$ , where  $I_1(x)$  and  $K_1(x)$  are the modified Bessel functions of the first order and  $C_0$  is an arbitrary constant to be determined by using order physico-mathematical consideration.

We thus find that the dual equations (24) and (25) can be recast into the form (24) and (27) respectively.



If we then set

$$\int_0^{\infty} (p^2/\beta^2 + \xi^2)^{1/2} D(\xi) J_1(\xi\rho) d\xi = g(\rho), (\rho > 1) \quad \dots(28)$$

and use the Hankel inversion formula to eqns. (24) and (28), we obtain that

$$(p^2/\beta^2 + \xi^2)^{1/2} D(\xi) = \frac{\alpha}{p} J_2(\xi) + \xi \int_1^{\infty} \lambda g(\lambda) J_1(\xi\lambda) d\lambda \quad \dots(29)$$

(See Gradshteyn and Ryzhik<sup>11</sup>, p. 683) and that eqn. (27) finally gives rise to the following integral equation for the unknown function  $g(\rho)$ :

$$\int_1^{\infty} \lambda g(\lambda) M(\rho, \lambda) d\lambda = C_0 K_1\left(\frac{p\rho}{\beta}\right) - \frac{\alpha}{p} \int_0^{\infty} \frac{J_2(\xi) J_1(\xi\rho)}{(\xi^2 + p^2/\beta^2)^{1/2}} d\xi, \quad (\rho > 1) \quad \dots(30)$$

where

$$M(\rho, \lambda) = \int_0^{\infty} \frac{\xi}{(\xi^2 + p^2/\beta^2)^{1/2}} J_1(\xi\rho) J_1(\xi\lambda) d\xi. \quad \dots(31)$$

The integral equation (30) is similar to the one obtained by Shail<sup>3</sup>, and can be attacked for solution, for large  $p$ , by a repeated use of the Abel's inversion formulae as described by Shail<sup>3</sup>, if the following asymptotic result for the kernel  $M(\rho, \lambda)$  is made use of:

$$M(\rho, \lambda) \sim \frac{1}{\pi(\rho\lambda)^{1/2}} e^{q(\rho+\lambda)} \int_{\max(\rho, \lambda)}^{\infty} \frac{e^{-2qw} dw}{(w-\rho)^{1/2} (w-\lambda)^{1/2}} + O(q^{-2}), \quad \dots(32)$$

with  $q = p/\beta$ .

We do not proceed any further with this problem here, as the other details are going to be repetitions of Shail's work<sup>3</sup>.

### Problem 3

The following dual integral equations arise, as shown by Goodrich<sup>4</sup>, in the study of a viscous flow problem induced by a rotating circular disc, kept on the surface of a bulk fluid of viscosity  $\mu$ , which is otherwise contaminated by an adsorbed fluid film of different viscosity  $\eta$ .

$$\int_0^{\infty} f(y) J_1(yr) dy = \omega r, (0 \leq r \leq a) \quad \dots(33)$$

$$\int_0^{\infty} (\mu y + \eta y^2) f(y) J_1(yr) dy = 0, (r > a) \quad \dots(34)$$

where  $\omega$  is the constant angular velocity of the disc of radius  $a$ , rotating around its axis.

Goodrich<sup>4</sup> has devised a special method of solving the above dual equations (33) and (34) in the three different circumstances, as given by the cases (i)  $\mu = 0$ , (ii)  $\eta = 0$  and (iii)  $\mu \neq 0$ ,  $\eta \neq 0$ . But, as pointed out by Shail<sup>5</sup>, Goodrich's solutions are not the correct ones, since they involve certain divergent integrals.

It is because of this major difficulty that Shail<sup>5</sup> has attacked the physical problem of Goodrich<sup>4</sup> and several other generalization of it<sup>6-8</sup>, by a method utilizing the Green's function technique.

We have shown below that the dual integral equations (33) and (34) can also be attacked for their solution in the three cases (i), (ii) and (iii) as considered by Goodrich<sup>4</sup>, in a straightforward manner as described for the Problems 1 and 2 above, and we thus infer that the use of the Green's function technique can be avoided here also.

*Case (i) :  $\mu = 0$* — In this special case, the dual equations (33) and (34) take up the forms :

$$\int_0^{\infty} f(y) J_1(yr) dy = \omega r \quad (0 \leq r \leq a) \quad \dots(35)$$

$$\int_0^{\infty} y^2 f(y) J_1(yr) dy = 0 \quad (r > a) \quad \dots(36)$$

and, the second eqn. (36), can be recast, by using the Bessel equation (1f), in the form

$$\int_0^{\infty} f(y) J_1(yr) dy = C_0/r \quad (r > a) \quad \dots(37)$$

where  $C_0$  is an arbitrary constant to be determined.

A straightforward use of the Hankel inversion formula to eqn. (35) and (37) gives

$$f(y) = C_0 J_0(ay) + \omega a^2 J_2(ay). \quad \dots(38)$$

(see Gradshteyn and Ryzhik<sup>11</sup>, p. 683).

We ultimately find that the constant  $C_0$  appearing in (38) must be chosen to be  $\omega a^2$  in order to make the integral in (36) convergent, in the sense mentioned in the



introduction, and this agrees with the observation of Goodrich, even though  $f(y)$  is different.

*Case (ii) :  $\eta = 0$ —* In this case, the dual equations to be solved are the ones as given by eqn. (35) and the new equation

$$\int_0^{\infty} y f(y) J_1(yr) dy = 0, (r > a). \quad \dots(39)$$

Assuming that

$$\int_0^{\infty} y f(y) J_1(yr) dy = g(r), (0 \leq r \leq a) \quad \dots(40)$$

and using Hankel's inversion formula we find that

$$f(y) = \int_0^{\infty} g(\lambda) \lambda J_1(\lambda y) d\lambda. \quad \dots(41)$$

Then using (41) in eqn. (35) we ultimately derive that

$$\frac{2}{\pi r} \int_0^r \frac{v^2 dv}{(r^2 - v^2)^{1/2}} \int_v^a \frac{g(\lambda) d\lambda}{(\lambda^2 - v^2)^{1/2}} = \omega r, (0 \leq r \leq a) \quad \dots(42)$$

obtained after utilizing the formula (10).

A repeated Abel inversion procedure, like the one adopted in the previous problems, ultimately gives

$$\pi g(r) = 4\omega r (a^2 - r^2)^{-1/2}, (0 \leq r \leq a) \quad \dots(43)$$

and the solution of the dual equations can be completed by using (41).

*Case (iii) :  $\mu \neq 0, \eta \neq 0$ —* In the most general case of the dual equations (33) and (34), when neither  $\mu$  nor  $\eta$  is zero, we obtain a Fredholm integral equation of the second kind, which is similar to the one obtained by Shail<sup>5</sup>, by means of a procedure as described below.

We first rewrite eqns. (33) and (34) in the form

$$\int_0^{\infty} f(y) J_1(yr) dy = \omega r (0 \leq r \leq a) \quad \dots(44)$$

and

$$\int_0^{\infty} y (1 + \lambda_0 ay) f(y) J_1(yr) dy = 0 (r > a) \quad \dots(45)$$

where  $\lambda_0 = \eta/\mu a$ .

Then, setting as in the previous problems,

$$\int_0^{\infty} y (1 + \lambda_0 ay) f(y) J_1(yr) dy = g(r) \quad (0 \leq r \leq a) \quad \dots(46)$$

and using Hankel's inversion formula to eqns. (46) and (45), we find that

$$f(y) = \frac{1}{(1 + \lambda_0 ay)} \int_0^a \lambda g(\lambda) J_1(y\lambda) d\lambda. \quad \dots(47)$$

We next use the relation (47) in (44) and interchange the orders of integration to obtain the following integral equation for the function  $g(r)$  :

$$\int_0^a \lambda g(\lambda) d\lambda \int_0^{\infty} \left( \frac{1}{1 + \lambda_0 ay} \right) J_1(yr) J_1(y) dy = \omega r, \quad (0 \leq r \leq a). \quad \dots(48)$$

Equation (48) is an integral equation of the first kind and it can be converted into an integral equation of the second kind, by observing first that

$$\frac{1}{1 + \lambda_0 ay} = \frac{1}{2} \left[ 1 + \frac{1 - \lambda_0 ay}{1 + \lambda_0 ay} \right] \quad \dots(49)$$

and then using the result (10) along with an Abel's inversion procedure, of a type similar to what is known as Williams's method<sup>12</sup>.

We find that the second kind Fredholm equation is obtained finally in the form

$$g^*(v) + \int_0^a L^*(t, v) g^*(t) dt = 4\omega v, \quad (0 \leq v \leq a) \quad \dots(50)$$

where

$$g^*(v) = v \int_v^a \frac{g(\lambda) d\lambda}{(\lambda^2 - v^2)^{1/2}}, \quad \dots(51)$$

and

$$L^*(t, v) = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \lambda_0 ay}{1 + \lambda_0 ay} \sin(ty) \sin(vy) dy. \quad \dots(52)$$

Equation (50) can be easily identified to be similar to the one obtained by Shail<sup>5</sup>, for the problem of Goodrich<sup>4</sup>.



A slightly more general dual integral equations, than the ones given by (33) and (34), which are

$$\int_0^{\infty} \left[ 1 + \frac{1 - \lambda_0 ay}{1 + \lambda_0 ay} e^{-2hy} \right] f(y) J_1(y\rho) dy = \omega\rho \quad (0 \leq \rho \leq a) \quad \dots(53)$$

and

$$2 \int_0^{\infty} f(y) y J_1(y\rho) dy = 0, \quad (\rho > a) \quad \dots(54)$$

where  $h > 0$ , can also be reduced to a Fredholm integral equation of the second kind by a method similar to the one described above.

The equations (53) and (54) arise in the rotating disc problem of Shail<sup>6</sup>, when the disc is kept at a distance  $a$  below the contaminated surface considered by Goodrich<sup>4</sup>, so that the particular case  $h = 0$  of (53) and (54) correspond to eqns. (33) and (34).

### 3. CONCLUSION

The present paper has dealt with the dual-integral-equations-formulation of some of the well-studied axi-symmetric mixed boundary value problems of potential theory in a manner different from the ones used in the literature, but is very straightforward otherwise. The principal aim has been to show that for the three problems discussed here, or its generalizations, it is not necessary to employ any other complicated technique, such as the Green's function technique, used by previous workers even though the merit of the latter technique is unquestionably of a higher level, if one has to handle axisymmetric bodies other than Circular discs as has been the case with the above three problems.

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## BANACH SPACE VALUED DISTRIBUTIONAL MELLIN TRANSFORM AND FORM INVARIANT LINEAR FILTERING

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From invariant filters are those shift variant filters such that a linear scaling of their input gives rise to a linear scaling of their output. In this work we develop a theory of Mellin transform and Mellin type convolution for Banach space valued distributions. Application of this theory to relate the input, output signals of the form invariant systems is given.

### 1. INTRODUCTION

Any physical system, be it electrical or optical, can be characterized by an input signal, output signal and system function. Time invariance (shift invariance) is a property possessed by many physical systems. Zemanian<sup>1,2</sup> introduced the concept of Banach space valued distributions. He presented a theory for the convolution and Laplace transformation of Banach space valued distributions and used these concepts for convolution representation of continuous linear time invariant systems.

The form invariant filters are those filters, such that a linear scaling of their input gives rise to a linear scaling (generally through a different scaling factor) of their output. Braccini and Gambardella<sup>3,4</sup> defined the form invariance property and obtained the most general class of linear shift variant systems satisfying it. They related the input and output signals of the form invariant systems by using Mellin transform. Further they discussed applications of form invariant filters in optical pattern recognition, image restoration, image reconstruction, radar signal processing etc. Braccini and Gambardella<sup>4</sup> considered the most general input output relationship for a linear one dimensional system as

$$g(x) = \int_{-\infty}^{\infty} f(t) w(t, x) dt$$

where  $f(t)$  is the input signal,  $g(x)$  is the output signal, and  $w(t, x)$  is the kernel of the integral transform. The integral representation has no sense when the input signal  $f(t)$  is a singularity signal. For example singularity signal like  $\delta$  cannot be treated as Lebesgue integrable function (LIF). If, we treat  $\delta$  as LIF we get senseless results (see Gelfand and Shilov<sup>5</sup>, p. 4, Zemanian<sup>6</sup>, p. 10). The theory of distributions (generalized functions) provides a natural language for such signals (see Zemanian<sup>7</sup>, Woblers and



Beltrami<sup>8</sup>). By using distribution theory we can generalize and correct results of Braccini and Gambardella<sup>4</sup>.

Recently the author<sup>9-11</sup> extended some integral transforms to Banach space valued distributions and discussed their applications to system theory.

In this paper we develop a theory of Mellin transform and Mellin type convolution for Banach space valued distributions. We use this theory to relate the input, output signals of the form invariant systems. Justification for these studies includes the desire to admit a larger class of systems and input output pairs. When the input signal  $f(t)$  is a LIF our results of section 5 reduce to results of Braccini and Gambardella<sup>4</sup>.

The plan of the paper is as follows. In section 3, we develop a theory for Mellin transform of Banach space valued distributions. In section 4 we present a theory for Mellin type convolution of Banach space valued distributions. Finally in section 5, we discuss Banach space valued distributional form invariant linear system and use Mellin transform to relate the input output signals of the form invariant systems.

## 2. NOTATIONS AND TERMINOLOGY

The notations and terminology of this work will follow that of Tiwari<sup>9,10,11</sup> and Zemanian<sup>1,2</sup>.  $\mathbb{R}$  and  $\mathbb{C}$  denote, respectively, the real line and complex plane.  $\mathbb{R}_+$  denotes the positive half-line  $\{t \in \mathbb{R} : 0 < t < \infty\}$ .  $A$  and  $B$  denote complex Banach spaces.  $E(A)$  denotes the linear space of all smooth functions  $\phi$  from  $\mathbb{R}$  into  $A$ .  $E_+(A)$  is linear space of all smooth functions from  $\mathbb{R}_+$  into  $A$ . Let  $K$  be a compact subset of  $\mathbb{R}$ .  $D_K(A)$  denotes the linear space of all smooth,  $A$  valued functions  $\phi$  such that  $\text{supp}(\phi) \subset K$ . The space  $D_K(A)$  is assigned the topology generated by the collection  $\{r_k\}_{k=0}^{\infty}$  of seminorms, where

$$r_k(\phi) = \sup_{t \in K} \|\phi^{(k)}(t)\|_A.$$

$\|\cdot\|_A$  denotes the norm in  $A$ . Let  $\{K_j\}_{j=1}^{\infty}$  be a sequence compact subsets in  $\mathbb{R}$  such that  $K_1 \subset K_2 \subset \dots$ ,  $\bigcup_{j=1}^{\infty} K_j = \mathbb{R}$ . The space  $D(A)$  is defined as the inductive limit space generated by  $D_{K_j}(A)$ . That is

$$D(A) = \text{ind}_{j \rightarrow \infty} D_{K_j}(A).$$

Similarly we have

$$D_+(A) = \text{ind}_{j \rightarrow \infty} D_{K_j}(A)$$

here  $K_1, K_2, \dots$ , are compact subsets of  $\mathbb{R}_+$  and  $\mathbb{R}_+ = \bigcup_{j=1}^{\infty} K_j$

For  $A = \mathbb{C}$ , we write  $E(A) = E$ ,  $E_+(A) = E_+$ ,  $D(A) = D$  and  $D_+(A) = D_+$ . When  $U$  and  $V$  are topological spaces,  $[U; V]$  denotes the space of all continuous linear mappings of  $U$  into  $V$ . The symbol  $\langle f, \phi \rangle$  denotes the element of  $V$  assigned to  $\phi \in U$  by  $f \in [U; V]$ . The notation  $\square$  denotes the end of a proof.

### 3. MELLIN TRANSFORMATION

Following Tiwari<sup>12</sup> we define the space  $M_{a,b,\alpha}(A)$ .

#### 3.1. The Space

$$M_{a,b,\alpha}(A)$$

For  $\alpha \geq 0$ , we define

$$M_{a,b,\alpha}(A) = \{\phi : \phi \in E_+(A),$$

$$i_{a,b,k}(\phi) = \sup_{\mathbb{R}_+} \|\lambda_{a,b}(t) t^{k+1} D_t^k \phi(t)\|$$

$$\leq C_k L^k k^{k\alpha}, k = 0, 1, \dots\}.$$

The constants  $L$  and  $C_k$  depend on the function  $\phi$  and

$$\lambda_{a,b}(t) = \begin{cases} t^{-a} & 0 < t \leq 1 \\ t^{-b} & 1 < t < \infty. \end{cases}$$

For  $k = 0$ , we set  $k^{k\alpha} = 1$ . The topology of the space  $M_{a,b,\alpha}(A)$  is generated by the family of seminorms  $\{i_{a,b,k}\}_{k=0}^{\infty}$ . The space  $M_{a,b}(A)$  is defined as the inductive limit space generated by  $M_{a,b,\alpha}(A)$ . That is

$$M_{a,b}(A) = \text{ind}_{\alpha \rightarrow \infty} M_{a,b,\alpha}(A).$$

Following Zemanian<sup>1</sup> (p. 102) we define the space  $M(w, z; A)$  as

$$M(w, z; A) = \text{ind}_{n \rightarrow \infty} M_{a_n, b_n}(A)$$

where

$$a_n \rightarrow w_+, b_n \rightarrow z_-, w, z \in [-\infty, \infty].$$

Let  $B$  be a complex Banach space. Any  $f \in [M(w, z; A), B]$  is a Banach space-valued distribution. When  $A = B = \mathbb{C}$ ,  $f$  becomes a scalar valued distribution.

Let  $S$  be a collection of bounded subsets of  $M_{a,b}(A)$ . The topology of uniform convergence on  $[M_{a,b}(A), B]$  is that generated by the collection of seminorms  $\{\sigma_{\Omega}\}_{\Omega \in S}$  where,

$$\sigma_{\Omega}(f) = \sup_{\phi \in \Omega} \|\langle f, \phi \rangle\|_B.$$



$$f \in [M_{a,b}(A), B], \Omega \in S.$$

The weak topology or simple topology for  $[M_{a,b}(A), B]$  is generated by the collection seminorms  $\{\rho_\phi\}$ , where  $\phi$  traverses  $M_{a,b}(A)$  and

$$\rho_\phi(f) = \| \langle f, \phi \rangle \|_B.$$

We now define Mellin transform of  $A$ -valued and  $[A, B]$  valued distributions (see Zemanian<sup>1</sup>, p. 115).

### 3.2. MELLIN TRANSFORM OF $A$ -VALUED DISTRIBUTIONS

Let  $f \in [D_+; A]$ . We say that  $f$  is Mellin transformable if there exists two members  $\sigma_1, \sigma_2 \in [-\infty, \infty]$  such that  $\sigma_1 < \sigma_2$ ,  $f \in [M(\sigma_1, \sigma_2); A]$ , and in addition  $f \notin [M(w, z); A]$  if either  $w < \sigma_1$  or  $z > \sigma_2$ . With  $\Omega_f = \{s : \sigma_1 < \operatorname{Re}(s) < \sigma_2\}$ ,  $t^{s-1} \in M(\sigma_1, \sigma_2)$  we define Mellin transform  $Mf$  of  $f$  as

$$F(s) = (Mf)(s) = \langle f(t), t^{s-1} \rangle, s \in \Omega_f. \quad \dots (3.1)$$

It can be easily proved that  $F(s)$  is an  $A$ -valued analytic function on  $\Omega_f$ .

The space  $[M(w, z; A); B]$  can be identified with the space  $[M(w, z); (A; B)]$  through the equation

$$\langle f_y, \phi \rangle_a = \langle y, \phi a \rangle$$

(see Zemanian<sup>1</sup>, p. 105) where  $f_y \in [M(w, z); [A; B]]$ ,

$$y \in [M(w, z; A); B], \phi \in M(w, z) \text{ and } a \in A.$$

Because of the above identification we use the same symbol to denote both  $f_y$  and  $y$ , and define Mellin transform of  $[A; B]$  valued distribution as :

We say that  $y \in [D_+; A]$  is Mellin transformable if there exists  $\eta_1, \eta_2 \in [-\infty, \infty]$  such that  $\eta_1 < \eta_2$ ,  $y \in [M(\eta_1, \eta_2; A); B]$  and  $y \notin [M(w, z; A); B]$  if either  $w < \eta_1$  or  $z > \eta_2$ .

Using the above identification  $y \in [M(\eta_1, \eta_2); [A; B]]$  also. Hence we now define the Mellin transform  $Y$  of  $y$  as

$$Y(s) = \langle y(t); t^{s-1} \rangle, s \in \Omega_y \quad \dots (3.2)$$

$$\text{where } \Omega_y = \{s : \eta_1 < \operatorname{Re}(s) < \eta_2\}, s \in \Omega_y \quad \dots (3.2)$$

where  $\Omega_y = \{s : \eta_1 < \operatorname{Re}(s) < \eta_2\}$  is called strip of definition for the Mellin transform of  $y$ .

**Theorem 3.1 (Analyticity Theorem)**—If  $My = Y(s)$  for  $s \in \Omega_y$ , then  $Y(s)$  is an  $[A; B]$  valued analytic function and for each non negative integer  $k$

$$Y^k(s) = \langle y(t), D_t^k t^{s-1} \rangle$$

PROOF : From definition the result is true for  $k = 0$ . With fixed  $s$  and  $\Delta s \neq 0$ , consider

$$\begin{aligned} & \langle y(t), \psi_{\Delta s}(t) \rangle \\ &= \frac{Y^k(s + \Delta s) - Y^k(s)}{\Delta s} = \langle y(t), D_s^k t^{s-1} \rangle. \end{aligned}$$

It is not difficult to show that  $\psi_{\Delta s}(t)$  converges to zero in  $M(\eta_1, \eta_2)$  as  $\Delta s \rightarrow 0$ .

#### 4. MELLIN TYPE CONVOLUTION

*Theorem 1.1*—The scaling operator  $S_a : \phi(t) \rightarrow \phi(at)$ ,  $a > 0$  is a topological automorphism on the space  $M_{a,b,\alpha}(A)$ .

PROOF :  $S_a$  is clearly well defined and linear. For continuity we observe that

$$\begin{aligned} & \sup_{\mathbf{R}_+} \| \lambda_{a,b}(t) t^{k+1} D_t^k \phi(at) \| \\ &= \sup_{\mathbf{R}_+} \| \lambda_{a,b}\left(\frac{T}{a}\right) \left(\frac{T}{a}\right)^{k+1} a^k D_T^k \phi(T) \|, T = at \\ &= C \sup_{\mathbf{R}_+} \| \lambda_{a,b}(T) T^{k+1} D_T^k \phi(T) \|, C \text{ is a constant} \\ &= C i_{a,b,k}(\phi). \end{aligned}$$

The inverse mapping  $S_a^{-1}$  is defined by  $S_a^{-1} : \phi(t) \rightarrow \phi\left(\frac{t}{a}\right)$ .

*Theorem 4.2*—If  $\phi \in M(w, z)$ , then, for each fixed  $t$ ,  $\phi(tx) \in M(W, z)$  as a function of  $x$ , where  $x > 0$ .

PROOF : Proof follows immediately from Theorem 4.1.

*Theorem 4.3*—If  $f$  is a member of  $[M(w, z); A]$  and  $\phi \in M(w, z)$ , then  $\phi \rightarrow \psi$  is a continuous linear mapping of  $M(w, z)$  into  $M(w, z; A)$ , where

$$\psi(t) = \langle f(x), \phi(tx) \rangle. \quad \dots(4.1)$$

PROOF : We first prove that  $\psi(t)$  is an  $A$ -valued smooth function on  $0 < t < \infty$ , i. e. we want to prove

$$\psi^k(t) = \langle f(x), D_t^k \phi(tx) \rangle. \quad \dots(4.2)$$

The above result is true for  $k = 0$  by definition.

For  $k = 1$ ,  $t$  fixed and  $\Delta t \neq 0$  consider

$$\frac{\psi(t + \Delta t) - \psi(t)}{\Delta t} = \langle f(x), D_t \phi(tx) \rangle = \langle f(x), \theta_{\Delta t}(x) \rangle \dots(4.3)$$



where

$$\theta_{\Delta t}(x) = \frac{1}{\Delta t} [\phi \{(t + \Delta t)x\} - \phi(t x)] - D_t \phi(t x).$$

We now show that  $\theta_{\Delta t}(x)$  converges in  $M(w, z)$  to zero as  $\Delta t \rightarrow 0$ . Using Taylor's formula and treating  $\Delta t$  as independent variable, it can be easily proved that

$$|\lambda_{a,b}(x) x^{p+1} \{\theta_{\Delta t}^p(x)\}| \rightarrow 0 \text{ as } \Delta t \rightarrow 0.$$

Thus result is established for  $k = 1$ . Now by induction the proof can be easily completed.

Next to prove  $\psi(t) \in M(w, z; A)$ , using the boundedness property of distributions we note that

$$\begin{aligned} & \|\lambda_{a,b}(t) t^{k+1} D_t^k \psi(t)\| \\ &= \|\lambda_{a,b}(t) t^{k+1} D_t^k \langle f(x), \phi(t x) \rangle\| \\ &\leq M \sup_{\mathbb{R}_+} |\lambda_{a,b}(t) t^{k+1} \lambda_{a,b}(x) x^{p+1} D_x^p D_t^k \phi(t x)| \\ &\leq M \sum_{\mathbb{R}_+} |\lambda_{a,b}(t x) (t x)^{p+k+1} D^{p+k} \phi(t x)| \\ &\leq M i_{a,b,p+k}(\phi). \end{aligned} \quad \dots(4.4)$$

Thus  $\psi(t) \in M(w, z; A)$ . Continuity of the mapping  $\phi \rightarrow \psi$  also follows from (4.4).

**Theorem 4.4**—The mapping  $f \rightarrow \psi$  defined by

$$\psi(t) = \langle f(x), \phi(t x) \rangle$$

is a linear mapping that is uniformly continuous with respect to  $S$  sets in  $M(w, z)$ .

**PROOF:** Let  $\Omega$  be any  $S$  set in  $M(w, z)$ . We choose  $a$  and  $b$  such that  $w < a < b < z$ . Then  $\Omega$  is a bounded set of  $M_{a,b}$ . Now  $\psi^k(t) = \langle f(x), \phi^k(t x) \rangle$  so that

$$\begin{aligned} & \lambda_{a,b}(x) x^{m+1} D_x^m [\lambda_{a,b}(t) t^{k+1} D_t^k \phi(t x)] \\ &= \lambda_{a,b}(t x) (t x)^{k+m+1} D^{k+m} \phi(t x). \end{aligned} \quad \dots(4.5)$$

From (4.5), for  $k$  fixed, as  $\phi$  traverses  $\Omega$  and  $t$  traverses  $\mathbb{R}_+$ ,  $\lambda_{a,b}(t) t^{k+1} D_t^k \phi(t x)$  as a function of  $x$  traverses a bounded set in  $\Theta$  in  $M_{a,b}$ . This means that  $\Theta$  is a  $S$  set in  $M(w, z)$ , and we have

$$\begin{aligned}
& \sup_{\phi \in \Omega} \sup_{t \in \mathbb{R}_+} \|\lambda_{a,b}(t) t^{k+1} \psi^k(t)\|_A \\
&= \sup_{\phi \in \Omega} \sup_{t \in \mathbb{R}_+} \|\lambda_{a,b}(t) t^{k+1} \langle f(x), \phi^k(tx) \rangle\|_A \\
&= \sup_{\theta \in \Theta} \|\langle f, \theta \rangle\|_A. \quad \dots(4.6)
\end{aligned}$$

Now if  $\Lambda$  is a neighbourhood of zero in  $M(w, z; A)$ ,  $(\Lambda) \cap M_{a,b}(A) \subset (\Lambda)$  is a neighbourhood of zero in  $M_{a,b}(A)$ . From (4.6), there exists a neighbourhood  $\Xi$  of zero in  $[M(w, z); A]$  such that  $\psi \in (\Lambda)$  for all  $\phi \in \Omega$  and all  $f \in \Xi$ .

We now define Mellin type convolution.

*Mellin Type Convolution*—Let  $a$  and  $b$  be two real numbers with  $a \leq b$ . Mellin type convolution is an operation that assigns to each arbitrary choice of the pair  $y \in [M(w, z; A); B]$  and  $f \in [M(w, z); A]$  the product  $y \vee f$  defined by

$$\begin{aligned}
\langle y \vee f, \phi \rangle &= \langle y(t), \langle f(x), \phi(tx) \rangle \rangle \\
&\quad \phi \in M(w, z) \quad \dots(4.7)
\end{aligned}$$

*Theorem 4.5*—Assuming the validity of Theorems 4.2, 4.3 and 4.4. The operator  $y \vee : f \rightarrow y \vee f$  is a continuous linear mapping of  $[M(w, z), A]$  into  $[M(w, z); B]$ .

**PROOF :** Linearity is clear from (4.7). To prove continuity we have to prove that any neighbourhood  $\Lambda$  of zero in  $[M(w, z); B]$  contain a neighbourhood  $\Xi$  of zero in  $[M(w, z); A]$ .

Let  $\Phi$  be an arbitrary  $s$  — set in  $M(w, z)$ . We have

$$\begin{aligned}
\sigma_{\Phi}(y \vee f) &= \sup_{\phi \in \Phi} \|\langle y \vee f, \phi \rangle\|_B \\
&= \sup_{\phi \in \Phi} \|\langle y, \psi \rangle\|_B.
\end{aligned}$$

Because  $y \in [M(w, z; A); B]$ , to each neighbourhood  $\Gamma$  of zero in  $B$  there exists a neighbourhood  $\Omega$  of zero in  $M(w, z; A)$  such that  $y$  maps  $\Omega$  into  $\Gamma$ . Now the proof can be completed by using the fact that  $y \vee f$  is a composite mapping  $\phi \rightarrow \psi \rightarrow \langle y, \psi \rangle$ .

We now prove in the following theorem that the operator  $y \vee$  commutes with the scaling operator  $S_a$ .

*Theorem 4.6*—If  $a > 0$ ,  $y \in [D_+(A) B]$  and  $f \in [E_+, A]$ , then

$$S_a(y \vee f) = y(S_a f).$$

**PROOF :** By Theorem 4.1  $S_a : \phi(t) \rightarrow \phi(at)$  is an automorphism on  $D_+$  and hence

$$\langle S_a(y \vee f), \phi(tx) \rangle = \langle (y \vee f), \frac{1}{a} \phi\left(\frac{tx}{a}\right) \rangle$$

(equation continued on p. 500)

$$= \langle y(t) \langle f(x), \frac{1}{a} \phi\left(\frac{tx}{a}\right) \rangle \rangle. \quad \dots(4.8)$$

Also

$$\begin{aligned} & \langle y \vee (S_a f), \phi(tx) \rangle \\ &= \langle y(t) \langle S_a f(x), \phi(tx) \rangle \rangle \\ &= \langle y(t) \langle f(x), \frac{1}{a} \phi\left(\frac{tx}{a}\right) \rangle \rangle. \quad \dots(4.9) \end{aligned}$$

From (4.8) and (4.9) our theorem is proved.

**Theorem 4.7**—If  $f \in [M_{a,b}(A), B]$  and  $w \in D_+(A)$ . Then,  $w \rightarrow g$  is a continuous linear mapping of  $D_+(A)$  into  $E_+(B)$ , where

$$g(x) = \langle f(t), \frac{1}{t} w\left(\frac{x}{t}\right) \rangle.$$

**PROOF :** It is easy to prove that  $g$  is smooth and the mapping is linear. To prove continuity we observe that

$$\begin{aligned} \|g^k(x)\|_B &= \|\langle f(t), D_x^k \left[ \frac{1}{t} w\left(\frac{x}{t}\right) \right] \rangle\| \\ &\leq \max_{0 \leq k \leq r} \sup_{\mathbb{R}_+} \|D_x^k D_t^1 \frac{1}{t} w\left(\frac{x}{t}\right)\|. \end{aligned}$$

From above continuity easily follows.

We call

$$(f \vee w)(x) = \langle f(t), \frac{1}{t} w\left(\frac{x}{t}\right) \rangle.$$

The Mellin type regularization of  $f$  by  $w$ .

**Theorem 4.8**—If  $y \in [D_+(A); B]$  and  $My = Y(s)$  for  $s \in \Omega_y$ , if  $f \in [D_+(A)]$  and  $Mf = F(s)$  for  $s \in \Omega_f$ , and if  $\Omega_y \cap \Omega_f$  is not empty then  $y \vee f$  exists in the sense of Mellin type. Convolution in  $[M(w, z), B]$  where the interval  $(w, z)$  is the intersection of  $\Omega_y \cap \Omega_f$  with the real axis. Moreover,

$$M(y \vee f) = Y(s) F(s).$$

**PROOF :** It is already proved in Theorem 4.5 that  $y \vee f \in [M(w, z); B]$ . Further because  $t^{s-1} \in M(w, z)$  for each fixed  $s$  with  $w < \operatorname{Re}(s) < z$ ,

$$\begin{aligned} M(y \vee f) &= \langle y(t), f(x), (tx)^{s-1} \rangle \\ &= \langle y(t), t^{s-1} \rangle \langle f(x), x^{s-1} \rangle \\ &= Y(s) F(s). \end{aligned}$$

Note that  $Y(s) F(s)$  is  $B$ -valued function for any fixed  $s \in \Omega_y \cap \Omega_f$ .



## 5. FORM INVARIANT LINEAR FILTERING

Generalizing the result of Braccini and Gambardella<sup>4</sup> (p. 1613) we write the input output relationship for a linear one dimensional system as

$$g(x) = \langle f(t), w(t, x) \rangle \quad \dots(5.1)$$

where  $f(t) \in [E(A), B]$ ,  $w(t, x) \in E(A)$  and  $g(x)$  is the output signal. When  $A = B = \mathbb{C}$  the Banach space valued distribution  $f$  becomes scalar valued distribution. Further if  $f(t)$  is a regular scalar valued distribution generated by locally integrable function  $f$ , we can write (5.1) as

$$g(x) = \int_{-\infty}^{\infty} f(t) w(t, x) dt. \quad \dots(5.2)$$

Further generalizing the definition of Braccini and Gambardella<sup>4</sup> we define the Banach space valued distributional form invariance property as below :

Let  $g_a(x)$  be the output of the system (5.1) when the input  $f(t)$  is replaced by  $f(at)$ ,  $a$  being a positive real number. We say that the system is Banach space valued distributional form invariant if and only if

$$g_a(x) = \alpha g(\beta x) \quad \dots(5.3)$$

where  $\alpha$  and  $\beta$  are real functions of  $a$ . From (5.1)

$$g_a(x) = \langle f(at), w(t, x) \rangle$$

and

$$\alpha g(\beta x) = \alpha \langle f(t), w(t, \beta x) \rangle.$$

From (5.3) we get

$$\langle f(at), w(t, x) \rangle = \alpha \langle f(t), w(t, \beta x) \rangle. \quad \dots(5.4)$$

Using the following property of distributions, namely

$$\langle f(at), \phi(t) \rangle = \langle f(t), \frac{1}{a} \phi\left(\frac{t}{a}\right) \rangle$$

we get from (5.4)

$$\frac{1}{a} \langle f(t), w\left(\frac{t}{a}, x\right) \rangle = \alpha \langle f(t), w(t, \beta x) \rangle.$$

The above equation is true if and only if

$$\frac{1}{a} w\left(\frac{t}{a}, x\right) = \alpha w(t, \beta x). \quad \dots(5.5)$$

General solution of this equation is (see Braccini and Gambardella<sup>4</sup>, p. 1613).

$$w(t, x) = x^{-\sigma} w_0(t/x^\sigma) \quad \dots(5.6)$$

$$\beta = a^{1/\sigma}, \quad \alpha = a^{(\sigma-1)/\sigma}, \quad x \neq 0$$

where  $\delta$  and  $\sigma$  are real numbers. In (5.6) choice of  $\sigma$  affects the scale factor  $\beta$  of the output signal  $\alpha g(\beta x)$ , whereas the choice of  $\delta$  (for any given value of  $\sigma$ ) affects the amplitude factor  $\alpha$  of the output  $\alpha g(\beta x)$ .

We now discuss some particular cases of (5.6) and relate the input output signals of the form invariant systems using Banach space valued distributional Mellin transform.

*Case 1*—Taking  $\delta = \sigma = 1$ ,  $\beta = a$ ,  $\alpha = 1$  in (5.6), we get

$$w(t, x) = x^{-1} w_0(t-x), x \neq 0. \quad \dots(5.7)$$

After some simple manipulation (5.7) can be written equivalently as

$$w(t, x) = t^{-1} w_0(x/t), t \neq 0. \quad \dots(5.8)$$

Note that taking  $\beta = a$  implies the same scaling factor in both the input  $f(at)$  and the output  $\alpha g(\beta x)$  of system (5.1). Further  $\alpha = 1$  means no amplitude gain changes are undergone by the output  $\alpha g(\beta x)$ , when the input is linearly scaled.

Now using (5.8) we write input output relationship from (5.1) as

$$g(x) = \langle f(t), t^{-1} w_0(x/t) \rangle = (f \vee w_0). \quad \dots(5.9)$$

Note that here  $f$  is a Banach space valued distribution and not an ordinary integrable function as in Braccini and Gambardella.

Taking Mellin transform of both sides of (5.9), we get

$$G(s) = F(s) \cdot W_0(s). \quad \dots(5.10)$$

(5.10) is similar to relationship relating to the Laplace transforms of the output and the input of a shift invariant (time invariant) systems (see Zemanian<sup>1</sup>). The result is in agreement with result of Braccini and Gambardella<sup>4</sup> (p. 1618) when  $f(t)$  is a Lebesgue integrable function.

To discuss next particular case of (5.6) we need

*Theorem 5.1*—The operator  $P : M_{a-r, b-r, \alpha}(A) \rightarrow M_{a, b, \alpha}(A)$  defined by  $P(\phi) = t^r \phi$  is an isomorphism from the space  $M_{a-r, b-r, \alpha}(A)$  onto  $M_{a, b, \alpha}(A)$  where  $r$  is a real number.

**PROOF :** Observe that

$$\begin{aligned} & \sup_{\mathbf{R}_+} \|\lambda_{a,b}(t) t^{k+1} D_t^{\bar{k}}(t^r \phi)\| \\ &= \sup_{\mathbf{R}_+} \|\lambda_{a,b}(t) t^{k+1} \sum_{p=0}^{\bar{k}} C_p t^{r-p} D_t^{k-p} \phi\| \end{aligned}$$

where  $C_p$  is some constant.

$$= \sup_{R_+} \left\| \sum_{p=0}^k C_p \lambda_{a-r, b-r} t^{k+1-p} D_t^{k-p} \phi \right\|$$

$$\leq \sum_{p=0}^k C_p i_{a-r, b-r, k-p}(\phi).$$

This proves the continuity of the map  $P$ . Linearity of the map is easily seen. The inverse mapping  $p^{-1}: M_{a,b,\alpha}(A) \rightarrow M_{a-r,b-r,\alpha}(A)$  is defined by  $p^{-1}(\phi) = t^{-r} \phi$ . The linearity and continuity of  $p^{-1}$  can also be proved similarly. Thus  $P$  is an isomorphism.

Case II—Taking  $w(t, x) = t^\eta x^\mu w_0(x/t)$  where  $\eta$  and  $\mu$  are arbitrary real numbers.

$$g(x) = \langle f(t), w(t, x) \rangle \text{ as}$$

$$g(x) = \langle f(t), t^{-1} t^{\eta+1} x^\mu w_0(x/t) \rangle$$

$$= [f(t) \vee t^{\eta+1} x^\mu w_0(x/t)].$$

Using Theorem 5.1 and some simple properties of Mellin transform (see Sneddon<sup>13</sup>, p. 270) we have

$$G(s) = F(s + \eta + \mu + 1) w_0(s + \mu).$$

This result is generalization of the result obtained by Braccini and Gambardella<sup>4</sup> (p. 1618).

### CONCLUDING REMARKS

In this paper we have applied Banach space valued distributional Mellin transform to 1-dimensional linear filters. In future we plan to extend these results to 2-dimensional linear filters. For this we will develop two dimensional Mellin transform and Laplace-Mellin (or Fourier-Mellin) transform. These transforms will be used to relate the input output signals. We will also prove that form invariance implies the operator has a convolution (Mellin type) representation.

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## A CYLINDRICAL WAVE-MAKER IN LIQUID OF FINITE DEPTH WITH AN INERTIAL SURFACE

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This paper is concerned with the generation of waves in a liquid of uniform finite depth with an inertial surface composed of uniformly distributed non-interacting floating particles, due to forced symmetric motion prescribed on the surface of an immersed circular vertical wave-maker. The techniques of Laplace transform in time and a suitable Weber transform in the radial co-ordinate are used to solve the problem. It is shown that if the inertial surface is too heavy, time-harmonic disturbance due to the wave-maker has only local influence on the liquid.

### 1. INTRODUCTION

The classical wave-maker problem for the case of deep water with a free surface was solved long back by Havelock<sup>2</sup> wherein the wave-maker is either in the form of a vertical plane or a cylinder with circular cross-section. Later Rhodes-Robinson<sup>7</sup> extended them to include the effect of surface tension at the free surface. Recently there has been a considerable interest in different problems concerning generation of waves in a liquid with an inertial surface composed of a thin but uniform distribution of disconnected particles (e.g. broken ice, floating mat on water). Rhodes-Robinson<sup>8</sup>, Mandal and Kundu<sup>4,5</sup>, Mandal<sup>3</sup> considered problems involving generation of waves at an inertial surface due to different types of sources with time dependent strengths submerged in a fluid of both infinite and uniform finite depths. Rhodes-Robinson<sup>8</sup> also pointed out briefly the method of solving the plane-vertical wave-maker problem in a liquid with an inertial surface by a suitable use of Green's integral theorem in the liquid region after taking Laplace transform in time. However the circular cylindrical wave-maker problem needs attention as it can, but not easily be solved by this method. Here we use a suitable Weber transform<sup>1</sup> in the radial coordinate after employing Laplace transform in time to solve the problem. The important time-harmonic case is considered and the inertial surface depression is calculated. It is observed that the time-harmonic wave-maker affects the inertial surface only locally if it is too heavy.

### 2. STATEMENT AND FORMULATION OF THE PROBLEM

We consider the motion under gravity in an ideal liquid of volume density  $\rho$  covered by an inertial surface composed of uniformly distributed floating particles of

area density  $\rho \epsilon$ .  $\epsilon = 0$  corresponds to a liquid with a free surface. On an immersed vertical circular cylindrical wave-maker, the normal fluid velocity is supposed to be prescribed which is both time and depth dependent. We choose a cylindrical coordinate system  $(r, \theta, y)$  in which the  $y$ -axis is taken as the axis of the cylinder with radius  $a$  so that  $r = a$  is the wave-maker  $0 < y < h$   $r > a$  is the fluid region and  $y = 0$ ,  $r > a$  is the position of the inertial surface at rest. The wave-maker starts operating from time  $t = 0$  with outward normal velocity  $U(y, t)$  on its boundary  $r = a$ . We consider only the axisymmetric case in which the resulting motion in the liquid is independent of  $\theta$ . Since the motion starts from rest it is irrotational and can be described by a velocity potential  $\phi(r, y, t)$  satisfying the Laplace's equation

$$\phi_{rrr} + \frac{1}{r} \phi_r + \phi_{yy} = 0, \quad r > a, \quad 0 < y < h. \quad \dots(2.1)$$

The condition at the inertial surface  $y = 0$  relating the potential function and the inertial surface depression  $\zeta$ , within the frame-work of linearised theory, consists of the kinematic condition

$$\phi_y = \zeta_t \quad \text{on } y = 0 \quad \dots(2.2)$$

and the dynamic condition

$$\phi_t = g \zeta + \epsilon \zeta_{tt} \quad \dots(2.3)$$

where  $g$  is the gravity. Elimination of  $\zeta$  produces the inertial surface condition

$$\Phi_{tt} - g \phi_y = 0 \quad \text{on } y = 0 \quad \dots(2.4)$$

where

$$\Phi = \phi - \epsilon \phi_y. \quad \dots(2.5)$$

The condition at the wave-maker is

$$\phi_r = U(y, t) \quad \text{on } r = a \quad \dots(2.6)$$

and the condition at the bottom is

$$\phi_y = 0 \quad \text{on } y = h. \quad \dots(2.7)$$

There are also initial conditions at the inertial surface given by

$$\frac{\partial \Phi}{\partial t} = \Phi = 0 \quad \text{on } y = 0 \quad \text{at } t = 0. \quad \dots(2.8)$$

Let a bar above a function denote its Laplace transform in time. Then  $\bar{\phi}(r, y; p)$  satisfies the boundary value problem

$$\begin{aligned} \bar{\phi}_{rrr} + \frac{1}{r} \bar{\phi}_r + \bar{\phi}_{yy} &= 0, \quad r > a, \quad 0 < y < h \\ p^2 \bar{\phi} - (g + \epsilon p^2) \bar{\phi}_y &= 0 \quad \text{on } y = 0, \quad r > a \end{aligned}$$



$$\begin{aligned}\bar{\varphi}_r &= \bar{U}(y, p) \quad \text{on } r = a \\ \bar{\varphi}_y &= 0 \quad \text{on } y = h.\end{aligned}\quad \dots(2.9)$$

## 3. SOLUTION BY WEBER-TRANSFORM METHOD

We use the following form of Weber-transform of a function  $f(r)$  defined in  $(a, \infty)$  by

$$g(\xi) = \int_a^\infty r A(r, \xi) f(r) dr \quad (\xi > 0) \quad \dots(3.1)$$

where

$$A(r, \xi) = J_1(a\xi) Y_0(r\xi) - J_0(r\xi) Y_1(a\xi) \quad \dots(3.2)$$

$J_n, Y_n$  ( $n = 0, 1$ ) are the Bessel functions of the first and second kinds respectively. (cf. Davies<sup>1</sup>, p. 252).

The inverse transform formula is

$$f(r) = \int_0^\infty \frac{\xi A(r, \xi)}{J_1^2(a\xi) + Y_1^2(a\xi)} g(\xi) d\xi. \quad \dots(3.3)$$

It may be noted that

$$\int_a^\infty \left( f_{rr} + \frac{1}{r} f_r \right) r A(r, \xi) dr = -\frac{2}{\pi\xi} f_r(a) - \xi^2 g(\xi). \quad \dots(3.4)$$

Let  $\psi(\xi, y; p)$  denote the Weber transform of  $\bar{\varphi}(r, y; p)$  as defined by (3.1). Then in view of (3.4),  $\psi(r, \xi)$  satisfies

$$\left. \begin{aligned}\psi_{yy} - \xi^2 \psi &= \frac{2}{\pi\xi} \bar{U}(y; p), \quad 0 < y < h \\ p^2 \psi - (g + \epsilon p^2) \psi_y &= 0 \quad \text{at } y = 0 \\ \psi_y &= 0 \quad \text{at } y = h.\end{aligned} \right\} \quad \dots(3.5)$$

The solution of (3.5) is

$$\psi(\xi, y; p) = -\frac{2}{\pi\xi} \int_0^h G(y, \alpha) \bar{U}(\alpha, p) d\alpha \quad \dots(3.6)$$

where  $G(y, \alpha)$  is the associated Green's function given by (cf. Mikhlin<sup>6</sup>)

$$G(y, \alpha) = \frac{\{p^2 \sinh \xi y + (g + \epsilon p^2) \xi \cosh \xi y\} \cosh \xi(h - \alpha)}{\xi \{p^2 \cosh \xi h + (g + \epsilon p^2) \xi \sinh \xi h\}} \quad \dots(3.7)$$

for  $0 < y < \alpha$ . For  $\alpha < y < h$ ,  $y$  and  $\alpha$  are to be inter-changed in the expression (3.7). Using the inverse Weber transform formula (3.3) we obtain

$$\bar{\varphi}(r, y, p) = -\frac{2}{\pi} \int_0^{\infty} \frac{A(r, \xi)}{J_1^2(a\xi) + Y_1^2(a\xi)} \int_0^h G(y, \alpha) \bar{U}(\alpha, p) d\alpha d\xi.$$

Expressing the Bessel functions of the first and second kinds in terms of the Hankel functions and rearranging  $G$  we obtain

$$\begin{aligned} \bar{\varphi}(r, y, p) = & -\frac{1}{\pi i} \int_0^{\infty} \frac{B(r, \xi)}{\xi D(\xi)} \left[ E(\xi, y) + \frac{\cosh \xi(h-y)}{\sinh \xi h} \frac{\mu^2}{\mu^2 + p^2} \right] \\ & \times \int_0^h \cosh \xi(h-\alpha) \bar{U}(\alpha, p) d\alpha d\xi \end{aligned} \quad \dots(3.8)$$

where

$$B(r, \xi) = \frac{H_0^{(1)}(\xi r)}{H_1^{(1)}(\xi a)} - \frac{H_0^{(2)}(\xi r)}{H_1^{(2)}(\xi a)}$$

and

$$\mu^2 = \frac{g \xi \sinh \xi h}{D(\xi)} \quad \dots(3.9)$$

$$D(\xi) = \cosh \xi h + \xi \epsilon \sinh \xi h$$

$$E(\xi, y) = \sinh \xi y + \xi \epsilon \cosh \xi y.$$

Taking Laplace's inversion we obtain

$$\begin{aligned} \varphi(r, y, t) = & -\frac{1}{\pi i} \int_0^{\infty} \frac{B(r, \xi)}{\xi D(\xi)} \int_0^h \left[ E U(\alpha, t) \right. \\ & \left. + \frac{\mu \cosh \xi(h-y)}{\sinh \xi h} \int_0^t U(\alpha, \tau) \sin \mu(t-\tau) d\tau \right] \\ & \times \cosh \xi(h-\alpha) d\alpha d\xi. \end{aligned} \quad \dots(3.10)$$

(3.10) gives the general result for the potential function due to a vertical cylindrical wave maker with prescribed time-dependent normal fluid velocity  $U(y, t)$ . The depression of the inertial surface at time  $t$  can be obtained from the relation

$$\zeta(r, t) = \frac{1}{g} \frac{\partial}{\partial t} (\varphi - \epsilon \varphi_y)(r, 0, t). \quad \dots(3.11)$$

## 4. TIME-HARMONIC WAVE-MAKER AND STEADY-STATE DEVELOPMENT

For a time-harmonic wave-maker we take

$$U(y, t) = U(y) \sin \sigma t \quad \dots(4.1)$$

where  $\sigma$  is the circular frequency. Then (3.10) gives

$$\begin{aligned} \varphi(r, y, t) = & -\frac{1}{\pi i} \int_0^h U(\alpha) \int_0^\infty \frac{B(r, \xi)}{\xi D(\xi)} \left[ E \sin \sigma t + \frac{\mu \cosh \xi (h-y)}{\sinh \xi h} \right. \\ & \times \left. \frac{\mu \sin \sigma t - \sigma \sin \mu t}{\mu^2 - \sigma^2} \right] \cosh \xi (h - \alpha) d\xi d\alpha. \end{aligned} \quad \dots(4.2)$$

To find the steady-state development in (4.2) we follow the method used by Rhodes-Robinson<sup>8</sup>. Two cases are required to be investigated according as the integrand of the inner integral in the second term of (4.2) has a pole in the range of integration  $\xi > 0$  or not. Now  $\mu^2 - \sigma^2$  or equivalently  $\xi \sinh \xi h - K^* \cosh \xi h$  has a zero at  $\xi = \xi_0$ , say, for  $\xi > 0$  if  $0 \leq \epsilon K < 1$  and none if  $\epsilon K \geq 1$  where  $K = \sigma^2/g$  and  $K^* = K(1 - \epsilon K)^{-1}$ . The latter case is physically interpreted as the inertial surface to be "too heavy" while the former as "light". The two cases are now dealt with separately.

For  $0 \leq \epsilon K < 1$ , we introduce a Cauchy principal value at  $\xi = \xi_0$  (i.e.  $\mu = \sigma$ ) and write the inner involving  $\sin \mu t$  in (4.2) as

$$\begin{aligned} & \sigma \int_0^\infty \frac{B(r, \xi) \cosh \xi (h-y) \cosh \xi (h-\alpha)}{\xi D(\xi) \sinh \xi h} \frac{\mu \sin \sigma t d\xi}{\mu^2 - \sigma^2} \\ &= 4\sigma \int_0^{(g/\epsilon)^{1/2}} \left[ \frac{\cosh \xi' (h-y) \cosh \xi' (h-\alpha) B(r, \xi')}{P(\xi') (\mu' + \sigma)} \right]_{\xi'=\xi_0}^{\xi} \frac{\sin \mu t}{\mu - \sigma} d\mu \\ &+ 2 \frac{B(r, \xi_0) \cosh \xi_0 (h-y) \cosh \xi_0 (h-\alpha)}{P(\xi_0)} \int_0^{(g/\epsilon)^{1/2}} \frac{\sin \mu t}{\mu - \sigma} d\mu \end{aligned}$$

where

$$P(\xi') = \sinh 2\xi' h + 2\xi' h \text{ and } \mu' = \mu(\xi').$$

By Riemann-Lebesgue lemma the first term is transient and the integral in the second term becomes  $\pi \cos \sigma t$  as  $t \rightarrow \infty$ . Thus as  $t \rightarrow \infty$ , we obtain after simplification

$$\varphi(r, y, t) \sim \frac{-\sin \sigma t}{\pi i} \int_0^h \int_0^\infty \frac{B(r, \xi)}{\xi} \frac{\{\xi \cosh \xi y - K^* \sinh \xi y\}}{\Delta(\xi)}$$

(equation continued on p. 510)



$$\begin{aligned} & \times \cosh \xi (h - \alpha) d\xi U(\alpha) d\alpha \\ & + \frac{2 \cos \sigma t}{i} \frac{B(r, \xi_0) \cosh \xi_0 (h - y) A_0}{P(\xi_0)} \end{aligned} \quad \dots(4.3)$$

where

$$\left. \begin{aligned} \Delta(\xi) &= \xi \sinh \xi h - K^* \cosh \xi h \\ A_0 &= \int_0^h \cosh \xi_0 (h - \alpha) U(\alpha) d\alpha. \end{aligned} \right\} \quad \dots(4.4)$$

(4.3) has the alternative representation

$$\begin{aligned} \varphi(r, y, t) \sim & - \frac{4 A_0 \cosh \xi_0 (h - y)}{P(\xi_0) H(\xi_0 a)} \{F(\xi_0 r) \cos \sigma t + G(\xi_0 r) \sin \sigma t\} \\ & - 4 \sin \sigma t \sum_{n=1}^{\infty} A_n \frac{K_0(\xi_n r)}{K_1(\xi_n a)} \frac{\cos \xi_n (h - y)}{\sin 2\xi_n h + 2\xi_n h} \end{aligned} \quad \dots(4.5)$$

where  $\xi_n$ 's are the solutions of the transcendental equation

$$\xi_n \sin \xi_n h + K^* \cos \xi_n h = 0, \quad n = 1, 2, \dots \quad \dots(4.6)$$

and

$$A_n = \int_0^h \cos \xi_n (h - \alpha) U(\alpha) d\alpha \quad \dots(4.7)$$

$$\left. \begin{aligned} F(\xi_0 r) &= J_0(\xi_0 r) J_1(\xi_0 a) + Y_0(\xi_0 r) Y_1(\xi_0 a) \\ G(\xi_0 r) &= J_0(\xi_0 r) Y_1(\xi_0 a) - Y_0(\xi_0 r) J_1(\xi_0 a) \\ H(\xi_0 a) &= J_1^2(\xi_0 a) + Y_1^2(\xi_0 a) \end{aligned} \right\} \quad \dots(4.8)$$

and  $K_0, K_1$  are modified Bessel's function. Hence using (3.11), the depression of the inertial surface as  $t \rightarrow \infty$  becomes

$$\begin{aligned} \zeta(r, t) \sim & - \frac{4\sigma}{g} (1 + \epsilon K^*) \cos \sigma t \sum_1^{\infty} A_n \frac{K_0(\xi_n r)}{K_1(\xi_n a)} \frac{\cos \xi_n h}{2\xi_n h + \sin 2\xi_n h} \\ & - \frac{4\sigma}{g} (1 - \epsilon K^*) \frac{A_0 \cosh \xi_0 h}{\sinh 2\xi_0 h + 2\xi_0 h} \frac{F \cos \sigma t - G \sin \sigma t}{H(\xi_0 h)}. \end{aligned} \quad \dots(4.9)$$

As  $r \rightarrow \infty$ , this gives

$$\zeta(r, t) \sim - \frac{4\sigma}{g} \frac{1 - \epsilon K^*}{H(\xi_0 a)} A_0 \left( \frac{2}{\pi \xi_0 r} \right)^{1/2}$$

(equation continued on p. 511)

$$\begin{aligned} & \times \left[ Y_1(\xi_0 a) \sin \left( \xi_0 r - \frac{\pi}{4} - \sigma t \right) \right. \\ & \left. + J_1(\xi_0 a) \cos \left( \xi_0 r - \frac{\pi}{4} - \sigma t \right) \right]. \end{aligned} \quad \dots(4.10)$$

(4.10) represents outgoing waves at large distance from the wave-maker.

For  $\epsilon K \geq 1$ , there is no pole of the integrand in the second term in (4.2) and thus by Riemann-Lebesgue lemma the integral involving  $\sin \mu t$  is wholly transient and hence  $t \rightarrow \infty$

$$\begin{aligned} \varphi(r, y, t) \sim & - \frac{\sin \sigma t}{\pi i} \int_0^h U(\alpha) \int_0^\infty \frac{B(r, \xi) \{ \xi \cosh \xi y + k_0 \sinh \xi y \}}{\xi (\xi \sin \xi h + k_0 \cosh \xi h)} \\ & \times \cosh \xi (h - \alpha) d\xi d\alpha \end{aligned} \quad \dots(4.11)$$

where

$$k_0 = K(\epsilon K - 1)^{-1}. \quad \dots(4.12)$$

This has the alternative representation

$$\begin{aligned} \varphi(r, y, t) \sim & - 4 \sin \sigma t \sum_{n=1}^{\infty} \frac{K_0(\zeta_n r)}{K_1(\zeta_n a)} \frac{\cos \zeta_n (h - y)}{2 \zeta_n h + \sin 2 \zeta_n h} \\ & \times \int_0^h \cos \zeta_n (h - \alpha) U(\alpha) d\alpha \end{aligned} \quad \dots(4.13)$$

where  $\zeta_n$ 's satisfy

$$\zeta_n \sin \zeta_n h - k_0 h \cos \zeta_n h = 0, \quad n = 1, 2, \dots \quad \dots(4.14)$$

Then the inertial surface depression as  $t \rightarrow \infty$  is

$$\begin{aligned} \zeta(r, t) \sim & \frac{4\sigma}{g(\epsilon K - 1)} \cos \sigma t \sum_1^{\infty} \frac{K_0(\xi_n r)}{K_1(\xi_n a)} \frac{\cos \zeta_n h}{2 \zeta_n h + \sin 2 \zeta_n h} \\ & \times \int_0^h \cos \zeta_n (h - \alpha) U(\alpha) d\alpha. \end{aligned} \quad \dots(4.15)$$

As  $r \rightarrow \infty$ ,  $\zeta(r, t) \rightarrow 0$ . Thus when the inertial surface is too heavy a time-harmonic disturbance on the wave-maker cannot propagate at large distances from the wave-maker.

## 5. CONCLUSION

The problem of a vertical circular cylindrical wave-maker immersed in a liquid of finite depth with an inertial surface is solved by the use of Laplace transform in

time and a suitable Weber transform in the radial co-ordinate. The steady-state development of the depression of the inertial surface due to a time-harmonic vertical circular cylindrical wave-maker is deduced for a 'light' as well as a 'heavy' inertial surface. In the absence of inertial surface, the results for a time-harmonic wave-maker are recovered which can also be deduced from Rhodes-Robinson's<sup>7</sup> results in the absence of surface tension.

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# NUMERICAL SOLUTION OF UNSTEADY FLOW AND HEAT TRANSFER IN A MICROPOLAR FLUID PAST A POROUS FLAT PLATE

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The problem of unsteady laminar flow and heat transfer in an incompressible micropolar fluid past an infinite porous flat plate has been examined. The flat plate is subjected initially to a constant suction velocity followed by a step function change for time  $t' > 0$ . An explicit finite difference scheme has been used to solve the governing equations of motion and energy. The velocity, microrotation and temperature distribution have been displayed through graphs for various values of micropolar parameter  $R$  at different time levels.

## 1. INTRODUCTION

Considerable attention has been paid by the researchers on the phenomenon of boundary layer flow over a permeable surface through which the fluid is either sucked or injected, because of its practical applications to boundary layer control and thermal protection in high energy flow by means of mass transfer. A class of unsteady solution of Navier-Stokes equations which possesses boundary layer character, is obtained when the velocity components are independent of longitudinal co-ordinate with free stream velocity either a function of time or constant<sup>1</sup>.

According to Stuart<sup>2</sup>, this special consideration leads to the exact solution of the flow equations for the arbitrary free stream velocity.

$$U(t) = U_0 [1 + f(t)] \quad \dots(1)$$

at a very large distance from the solid boundary. Watson<sup>3</sup> investigated the plane flows with special forms of function  $f(t)$  in the arbitrary external velocity. Further  $f(t) = 0$  in the free stream velocity leads to a simple solution to the plane flows. Using this consideration, the problem of unsteady flow with step function change in suction velocity have been analysed in Newtonian fluid<sup>4</sup>.

The general theory of micropolar fluid which has been formulated and presented by Eringen<sup>5,6</sup> is deviating from that of Newtonian fluid by accomodating two new variables viz. microrotation i.e. spin and micro inertia describing the distributions of atoms and molecules inside the fluid elements. Peddieson and McNitt<sup>7</sup> and Wilson<sup>8</sup> first developed the boundary layer concept for such a fluid and investigated the flow past a flat plate. Dey and Nath<sup>9</sup> studied micropolar fluid flow over a semi-infinite plate using parabolic co-ordinates to consider the flow regime including the leading edge. Gorla *et al.*<sup>10</sup> solved the steady state heat transfer in a micropolar fluid flow over a semi-infinite plate using similarity variables. Chawla<sup>11</sup> obtained the solution for unsteady micropolar fluid flow past an infinite plate.

In the present study, the effect of step function change in suction velocity on the flow and heat transfer in an incompressible micropolar fluid past a flat plate has been studied by considering the free stream velocity to be constant for all time  $t' \geq 0$ . At time  $t' = 0$ , we assume that there is a steady flow over the plate with constant suction  $v'_1 < 0$ . For time  $t' > 0$ , a step function change is made in suction velocity which is responsible for unsteadyness of the flow and heat transfer. The velocity and temperature are assumed to be functions of transverse co-ordinate  $y'$  and time  $t'$  only. An explicit finite difference scheme is employed to solve the governing equations.

## 2. MATHEMATICAL FORMULATION

The unsteady two-dimensional flow of an incompressible micropolar fluid past a semi-infinite porous flat plate is considered. The  $X'$ -axis is chosen along the plate with leading edge as the origin and  $Y'$ -axis, at right angles to it. Let  $u'$  and  $v'$  be the velocity components parallel and normal to the plate respectively. All the fluid properties are assumed to be constant. At time  $t' = 0$ , the flow of the fluid is assumed to be steady with constant suction velocity  $v'_1 < 0$ , normal to the plate. At time  $t' > 0$ , the suction velocity is suddenly changed into  $v'_2$  ( $v'_2 < 0$  for suction,  $v'_2 > 0$  for injection) which causes the flow unsteady. At all times, the fluid free stream velocity is assumed to be constant and paralalled to the plate. Since the resulting flow is superimposed and weak, and the plate is of infinite extent, concerned dependent variables may be considered as function of  $y'$  and  $t'$  only.

The governing equations of momentum and energy for heat conducting micropolar fluid<sup>6,12</sup> are

$$\frac{\partial v'}{\partial y'} = 0 \quad \dots(2)$$

$$\rho \left[ \frac{\partial u'}{\partial t'} + v' \frac{\partial u'}{\partial y'} \right] = (\mu + K) \frac{\partial^2 u'}{\partial y'^2} + K \frac{\partial N'}{\partial y'} \quad \dots(3)$$

$$0 = \frac{\partial P'}{\partial y'} \quad \dots(4)$$

$$\rho j \left[ \frac{\partial N'}{\partial t} + v' \frac{\partial N'}{\partial y'} \right] = \gamma \frac{\partial^2 N'}{\partial y'^2} - K \left[ \frac{\partial u'}{\partial y'} + 2N' \right] \quad \dots(5)$$

$$U(t) = U_{\infty} \text{ (constant)} \quad \dots(6)$$

$$\begin{aligned} \rho c_p \left[ \frac{\partial T}{\partial t} + v' \frac{\partial T}{\partial y'} \right] &= k_f \frac{\partial^2 T}{\partial y'^2} + \left( \mu + \frac{K}{2} \right) \left( \frac{\partial u'}{\partial y'} \right)^2 \\ &+ \frac{K}{2} \left[ \frac{\partial u'}{\partial y'} + 2N' \right]^2 + \gamma \left[ \frac{\partial N'}{\partial y'} \right]^2 \quad \dots(7) \end{aligned}$$

where  $\rho$  is the density,  $\mu$  the coefficient of viscosity,  $c_p$  the specific heat at constant pressure,  $T$  the temperature,  $\gamma$  and  $K$  are the micropolar material constant and  $k_f$  the thermal conductivity. In view of (1), we take

$$\begin{aligned} \frac{v'_2}{|v'_1|} &= -1, \text{ for } t' = 0 \\ &= -\lambda, \text{ for } t' > 0 \end{aligned} \quad \dots(8)$$

where  $\lambda$  is the suction parameter.

Introducing the following non-dimensional variables

$$\begin{aligned} y &= \frac{|v'_1| y'}{(\mu + K)}, \quad t = \frac{|v'_1|^2 t'}{(\mu + K)}, \quad u = \frac{u'}{u_{\infty}} \\ N' &= \frac{N(\mu + K)}{\rho u_{\infty} |v'_1|}, \quad \theta = (T - T_{\infty}) / (T_w - T_{\infty}) \end{aligned}$$

where  $T_w$  is the temperature at the isothermal wall and  $T_{\infty}$  the free stream temperature. Equations (3), (5) and (7) can be written as

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + \frac{R}{1+R} \frac{\partial N}{\partial y} \quad \dots(9)$$

$$\frac{\partial N}{\partial t} - \lambda \frac{\partial N}{\partial y} = \frac{A}{1+R} \frac{\partial^2 N}{\partial y^2} - \frac{2R(1+R)}{R_e^2} \left( N + \frac{1}{2} \frac{\partial u}{\partial y} \right) \quad \dots(10)$$

$$\frac{\partial \theta}{\partial t} - \lambda \frac{\partial \theta}{\partial y} = \frac{1}{Pr(1+R)} \frac{\partial^2 \theta}{\partial y^2}$$

$$+ E \left[ \frac{AR_e^2}{(1+R)^3} \left( \frac{\partial N}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \frac{2R}{1+R} \left( N^2 + N \frac{\partial u}{\partial y} \right) \right] \quad \dots(11)$$



where  $R (= K/\mu)$ ,  $A (= \gamma/\mu j)$  are the micropolar parameters,  $Pr (= \mu c_p/k_f)$  the Prandtl number,  $E (= u_\infty^2/c_p (T_w - T_\infty))$  the Eckert number and  $Re (= \rho |v_1| j^{1/2})/\mu$  is a dimensionless constant. The initial conditions correspond to the solution of the steady state problem.

The boundary conditions can be written as

$$\begin{aligned} u(0, t) = 0, \quad N(0, t) = 0.0, \quad \theta(0, t) = 1, \quad \text{at } y = 0, \\ u(\infty, t) = 1, \quad N(\infty, t) = 0.0, \quad \theta(\infty, t) = 0, \quad \text{at } y \rightarrow \infty. \end{aligned} \quad \dots(12)$$

### 3. METHOD OF SOLUTION

To solve the eqns. (9–11) subject to (12), the explicit finite difference scheme has been employed. As in Soundalgekar<sup>14</sup>, the boundary conditions at  $y = \infty$ , have been taken to be satisfied approximately at  $y = 6.0$ . The flow regime defined as a semi-infinite strip in time, bounded by  $y = 0$  and  $y = 6.0$  is divided into a grid by lines parallel to  $y$  and  $t$  axes (Fig. 1). The length step  $\Delta y$  is chosen as 0.1 while the time step  $\Delta t$  is taken to be 0.0025 so as to satisfy the stability condition as per Ralston<sup>15</sup>.

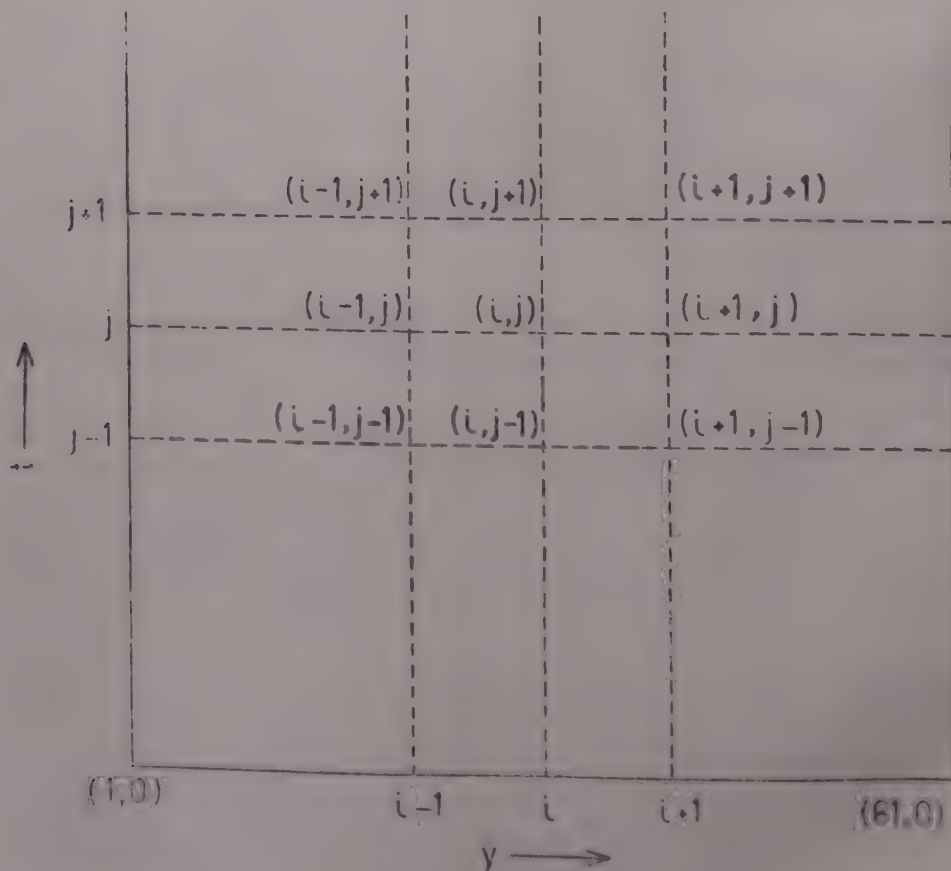


FIG. 1.

Replacing the time derivatives by forward difference and the space derivatives by their corresponding central differences, eqns. (9) -- (11) can be re-written as

$$u_i^* = u_i + \Delta t \left[ \left( \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2} \right) + \lambda \frac{u_{i+1} - u_{i-1}}{2\Delta y} + \frac{R}{(1+R)} \frac{(N_{i+1} - N_{i-1})}{2\Delta y} \right] \quad \dots(13)$$

$$N_i^* = N_i + \Delta t \left[ \frac{A}{(1+R)} \frac{(N_{i+1} - 2N_i + N_{i-1}))}{(\Delta y)^2} + \lambda \frac{(N_{i+1} - N_{i-1}))}{2\Delta y} - \frac{2R(1+R)}{R_e^2} \left( N_i + \frac{1}{2} \frac{u_{i+1} - u_{i-1}}{2\Delta y} \right) \right] \quad \dots(14)$$

$$\theta_i^* = \theta_i + \Delta t \left\{ \frac{1}{Pr(1+R)} \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta y)^2} + \lambda \frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} + E \left[ \frac{AR_e^2}{(1+R)^3} \left( \frac{N_{i+1} - N_{i-1}}{2\Delta y} \right)^2 + \left( \frac{u_{i+1} - u_{i-1}}{2\Delta y} \right)^2 + \frac{2R}{(1+R)} \left( N_i^2 + N_i \frac{u_{i+1} - u_{i-1}}{2\Delta y} \right) \right] \right\} \quad \dots(15)$$

where  $2 \leq i \leq 60$ .

The initial condition is taken as the steady state solution obtained by solving the corresponding steady state finite difference equations through Gauss Seidal method, while the boundary conditions (12) transform as

$$\begin{aligned} u_1 &= 0.0, u_{61} = 1.0 \\ N_1 &= 0.0, N_{61} = 0.0 \\ \theta_1 &= 1.0, \theta_{61} = 0.0. \end{aligned} \quad \dots(16)$$

The solution of the difference equations is obtained at the intersection of the grid lines, called nodes. In Fig. 1, the nodes are specified by double subscripts ( $i, j$ ) with the origin located at the intersection of the lines  $y = 0$  and  $t = 0$ . The value of the dependent variables  $u$ ,  $N$  and  $\theta$  at the nodal points along the lines  $y = 0$  and  $y = 6.0$  are known for all time  $t$ , while unknown values of  $u_i^*$  ( $= u_i^{j+1}$ ),  $N_i^*$  ( $= N_i^{j+1}$ ) and  $\theta_i^*$  ( $= \theta_i^{j+1}$ ), at internal nodes between  $y = 0$  and  $y = 6.0$  for any time  $t > 0$  are to be determined.

The non-dimensional shear stress, couple stress and the Nusselt number on the wall are respectively given by

$$C_f = \frac{t x' y'}{\rho u_\infty |v'_1|} = \left( \frac{\partial u}{\partial y} \right)_{y=0} \quad \dots(17)$$

$$C_m = \frac{m_{y'z'}}{(\rho^2 u_\infty |v'_1|^2 \gamma) k^2} = \frac{R^2}{(1+R)^2} \left( \frac{\partial N}{\partial y} \right)_{y=0} \quad \dots(18)$$

and

$$Nu^* = \frac{\mu Nu}{(L |v'_1| \rho)} = - \left( \frac{\partial \theta}{\partial y} \right)_{y=0} \quad \dots(19)$$

It has been observed that by increasing the micropolar effects in the fluid, more and more heat is transferred from the plate at any time level.

#### 4. NUMERICAL RESULTS AND DISCUSSION

Fixing  $A = 1.5$ ,  $Re = 0.5$ ,  $Pr = 1.0$ ,  $E = 0.1$  and  $\lambda = 2.0$ , the discussion is limited to the variation of micropolar parameter  $R$  for different time levels viz.  $t=0.1$ ,  $0.5$  and  $1.0$ .

In Fig. 2, as compared to the Newtonian fluid  $R = 0.0$ , the presence of micropolar additives in the fluid increases the velocity throughout. Also as time grows, the velocity profiles are further accelerated. The microrotation  $N$  is found to have negative

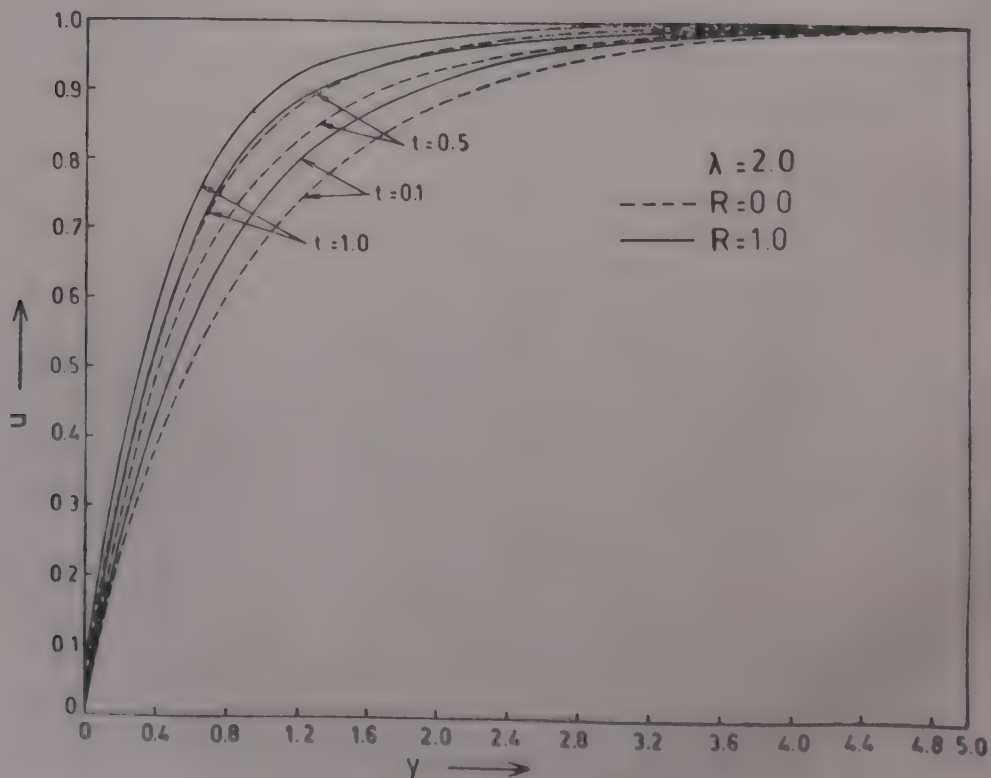


FIG. 2. Velocity distribution for different  $R$  with  $t$ .



values for all the parameters under study. Numerically the microrotation increases in a region near the plate reaching a maxima with increase in  $R$  as well as  $t$  (see Fig. 3). In Fig. 4, the temperature profiles are drawn. The temperature decreases with increase in  $R$  at all time levels having a square shape profile for  $R = 5$ .

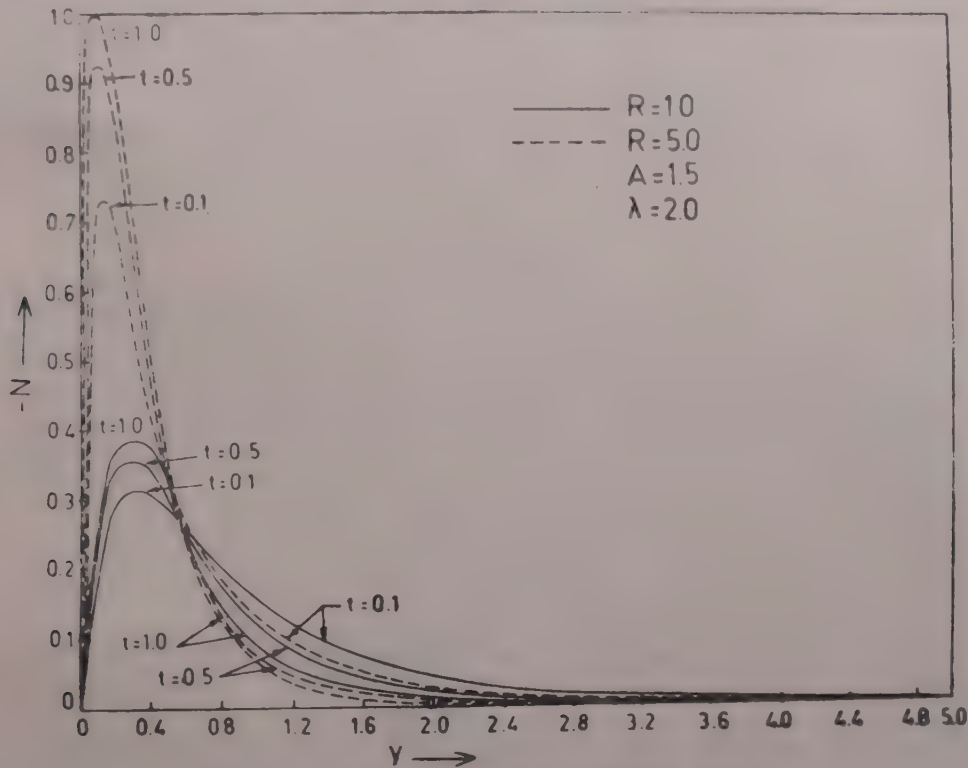


FIG. 3. Microrotation for different  $R$  and  $t$ .

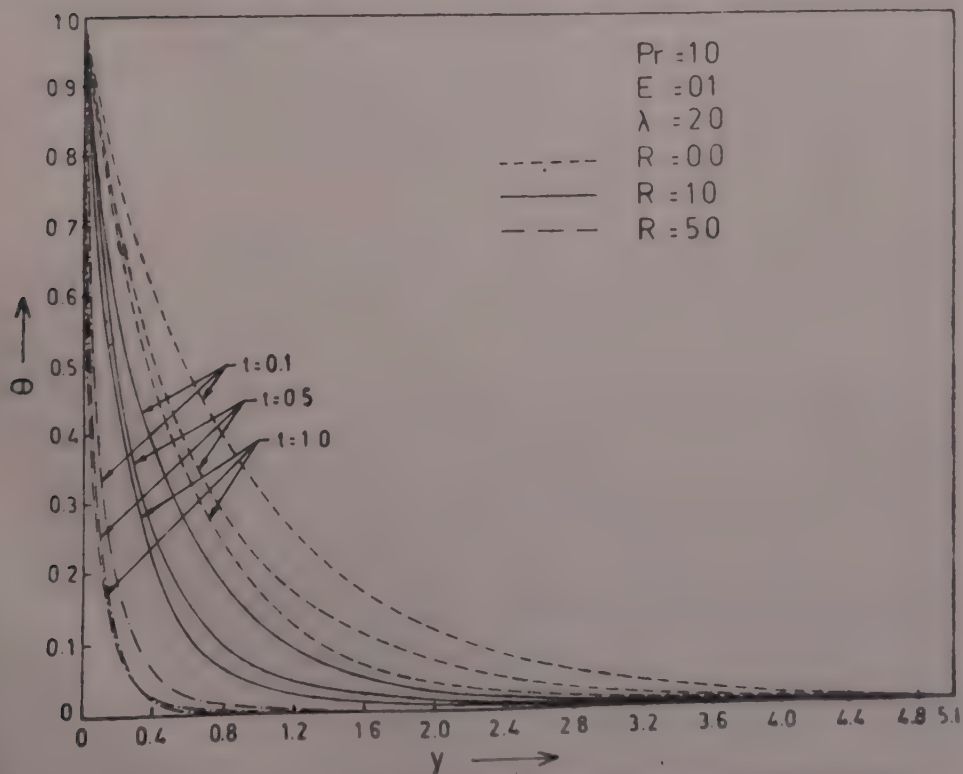


FIG. 4. Temperature distribution for various values of  $R$  with time  $t$ .

The difference scheme (13) – (15) is consistent as per condition stated in Ralston<sup>15</sup> and Rosenberg<sup>16</sup> with the differential equations (9) – (11), which in turn implies that the difference equations actually do approach the governing differential equations. The truncation error for the approximation in the velocity, microrotation and temperature is  $[O(\Delta t) + O(\Delta y)^2]$  which tends to become zero as  $\Delta y$  and  $\Delta t$  tend to zero. The programme has also been executed for smaller values of  $\Delta t$ , viz.  $\Delta t = 0.002, 0.001$  and  $0.0005$  and the results compared with those for  $\Delta t = 0.0025$ , reveal no significant change, thus ensuring the convergence of finite difference scheme employed here.

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## TORSIONAL VIBRATION OF A RANDOM ELASTIC CYLINDER

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We consider a random eigen-value problem arising from torsional vibrations of a cylinder with randomly varying density. The problem is transformed into a perturbed Fourier-Bessel equation and the perturbation technique is used to obtain its solution. An expression for variance of the eigen-value is also derived.

### INTRODUCTION

We consider the equation

$$\mu \left\{ \frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \right\} = \rho(r, \tau) \frac{\partial^2 u_\theta}{\partial t^2} \quad \dots(1)$$

where  $\mu$  is the shear modulus,  $\rho$  the density and  $r, \theta, z$  are the cylindrical polar coordinates. This equation arises from torsional vibrations of an elastic cylinder. The displacement of the  $z$ -axis is assumed to be zero so that

$$u_\theta = 0, \text{ when } r = 0 \text{ for all } z, t. \quad \dots(2)$$

Also the lateral surfaces of the cylinder are stress free. This gives rise to the boundary condition

$$\sigma_{r\theta} = \mu \left\{ \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right\} = 0 \text{ at } r = a \quad \dots(3)$$

for all  $z$  and  $t$ .

This problem has been studied by Achenbach<sup>1</sup> when the cylinder is hollow and Heath and Wood<sup>3</sup> when the rigidity of the cylinder varies radially. In this paper we use perturbation technique to study this problem when the density is a function of  $r$  and a random parameter  $\tau$ . This problem is of interest due to the fact that many new composite materials are effectively described as materials with random properties. In an earlier paper Zaman<sup>8</sup> has studied a random eigen-value problem arising from vibration of random elastic plates using the variational formulation. However, in this case we use perturbation method as the variational method does not seem to be applicable.

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## FORMULATION OF THE PROBLEM

We seek sinusoidal solutions of (1) and put

$$u_\theta = V(r) \exp [i(\omega t - \alpha z)] \quad \dots(4)$$

where  $\omega$  is the angular frequency and  $\alpha$  is the reduced wave number. The equation of motion (1) and the boundary conditions (2) and (3) transform into

$$V''(r) + \frac{1}{r} V'(r) + \left[ \frac{\rho(r, \tau)}{\mu} \omega^2 - \alpha^2 - \frac{1}{r^2} \right] V(r) = 0 \quad \dots(5)$$

with

$$\left. \begin{aligned} V(0) &= 0 \\ V'(a) &= \frac{1}{a} V(a) \end{aligned} \right\} \quad \dots(6)$$

where the time dependence  $e^{i\omega t}$  is understood and omitted throughout. Assuming that the density  $\rho$  can be written as

$$\rho(r, \tau) = (\rho_0 + \epsilon \rho_1(r, \tau)) \quad \dots(7)$$

where  $\rho_0$  is the mean of  $\rho$  and  $\rho_1$  is a random function with zero mean and  $\epsilon$  is a small parameter. Equation (5) can thus be written as

$$V''(r) + \frac{1}{r} V'(r) + \left[ \gamma - \epsilon v(r) - \frac{1}{r^2} \right] V(r) = 0 \quad \dots(8)$$

where

$$v(r) = \frac{-\rho_0 \rho_1(r, \tau) \omega^2}{\mu}$$

and

$$\gamma = \frac{\rho_0 \omega^2}{\mu} - \alpha^2.$$

Equation (8) is a perturbation of Bessel's equation of order 1. The boundary conditions (6) and eqn. (8) form the perturbed random boundary value problem. Neither the perturbed nor the unperturbed differential expression is formally self-adjoint and both have a singularity at  $r = 0$ .

## SOLUTION OF THE PROBLEM

We transform (8) into Liouville's normal form by putting

$$V(r) = \exp \left[ -\frac{1}{2} \int \frac{1}{t} dt \right] u(r). \quad \dots(9)$$

Equation (8) thus becomes

$$u''(r) + \left[ \gamma - \frac{3}{4r^2} - \epsilon v(r) \right] u(r) = 0. \quad \dots(10)$$

When  $\epsilon = 0$ , equation (10) reduces to the Fourier-Bessel equation with solutions

$$r^{1/2} J_1(kr), r^{1/2} Y_1(kr), \text{ where}$$

$$k = \left( \frac{\rho_0 \omega^2}{\mu} - \alpha^2 \right)^{1/2}$$

which is real and positive whenever  $r$  is real and positive (Titchmarsh<sup>6</sup>; p. 81).

The transformation (9) transforms the boundary conditions (6) into

$$\left. \begin{aligned} u(0) &= 0 \text{ at } r = 0, \\ u'(a) &= \frac{3}{2} u(a) \text{ at } r = a. \end{aligned} \right\} \quad \dots(11)$$

The problem (10) with  $\epsilon = 0$  is now self-adjoint with one singular end-point at  $r = 0$  where it is the limit point case<sup>6</sup>. It has a countable number of real simple eigen-values  $\gamma_0, \gamma_1, \gamma_2, \dots$  where  $\gamma_0 = 0$  and  $J_1(k_n a) = 0$  for  $n = 1, 2, \dots$

The associated eigen-functions are

$$\left. \begin{aligned} u_0(r) &= A_0 r^{3/2} \\ u_n(r) &= A_n r^{1/2} J_1(k_n r), \quad n > 1. \end{aligned} \right\} \quad \dots(12)$$

With a suitable choice of  $A_n$ ,  $\{u_n\}_{n=0}^{\infty}$  is an orthonormal set in  $L^2(0, a)$ .

We now write

$$U_n(r) = u_n(r) + \epsilon u_n^{(1)}(r) + \epsilon^2 u_n^{(2)}(r) + \dots \quad \dots(13)$$

$$A_n = \gamma_n + \epsilon \gamma_n^{(1)} + \epsilon \gamma_n^{(2)} \dots \quad \dots(14)$$

where  $u_n(r)$  and  $\gamma_n$  are the  $n$ th eigen-function and eigen-value of the unperturbed problem and are given by (12). Following Titchmarsh<sup>6</sup>, we write  $u_n^{(1)}, u_n^{(2)}$  as expansions in terms of the eigen-functions of the unperturbed problem as

$$\begin{aligned} u_n^{(1)}(r) &= \sum_{p=0}^{\infty} \alpha_{np} u_p(r) \\ u_n^{(2)}(r) &= \sum_{p=0}^{\infty} \beta_{np} u_p(r). \end{aligned} \quad \dots(15)$$

Using (13), (14) and (15) in the perturbed equation (10) and equating the coefficient of  $\epsilon$  to zero, we get

$$\begin{aligned} u_n^{(1)''}(r) + \left( \gamma_n - \frac{3}{4r^2} \right) u_n^{(1)}(r) + (\gamma_n^{(1)} - v(r)) \\ \times u_n(r) = 0. \end{aligned} \quad \dots(16)$$

We multiply equation (16) by  $u_n(r)$  and integrate using the orthonormality of  $u_n(r)$ 's to get

$$\gamma_n^{(1)} = -\frac{\rho_0 \omega^2}{\mu} \int_0^a \rho_1(r, \tau) u_n^2(r) dr \quad \dots(17)$$

where we have substituted value of  $v(r)$  introduced earlier. In a similar way, we multiply (16) by  $u_m(r)$  and integrate to get

$$\alpha_{nm} = \frac{-1}{\gamma_n - \gamma_m} \left[ \frac{\rho_0 \omega^2}{\mu} \int_0^a \rho_1(r, \tau) u_n(r) u_m(r) dr + \int_0^a \gamma_n^{(1)} u_n(r) u_m(r) dr \right] \quad \dots(18)$$

where  $\gamma_n^{(1)}$  is given by (17).

#### ESTIMATE ON THE VARIANCE

We note that  $\gamma_n^{(1)}$  is a weakly correlated random process (Boyce<sup>2</sup>). It is of the form

$$\gamma_n^{(1)} = \int_0^a f(r) P(r, \tau) dr \quad \dots(19)$$

where

$$f(r) = \frac{\rho_0 \omega^2}{\mu} u_n^2(r)$$

and

$$P(r, \tau) = \rho_1(r, \tau). \quad \dots(20)$$

The mean square of such a process is given by

$$\langle \gamma_n^{(1)2} \rangle = \int_0^a \int_0^a K(r_1, r_2) f(r_1) f(r_2) dr_1 dr_2 \quad \dots(21)$$

where  $K(r_1, r_2)$  is the correlation function for the random quantity  $\rho_1(r, \tau)$ . There are well documented experimental methods for determination of the correlation function appearing in (21) (Corson<sup>4</sup> Miller<sup>5</sup>). Zaman<sup>7</sup> has given an elementary method to derive the correlation function for a statistically homogeneous two phase composite. For two phase composites,  $K(r_1, r_2)$  is given by



$$K(r_1, r_2) = \rho_{1,1}^2 P_{11}(A, B) + \rho_{1,1} \rho_{1,2} P_{12}(A, B) \\ + \rho_{1,2} \rho_{1,1} P_{21}(A, B) + \rho_{2,2}^2 P_{22}(A, B) \quad \dots(22)$$

where the property  $\rho_1$  is related at two points  $A$  and  $B$  lying anywhere in the material.  $P_{ij}(A, B)$  is the probability that point  $A$  is in phase  $i$  and point  $B$  is in the phase  $j$  of the material and  $\rho_{1,j}$  is the value of  $\rho_1$  in the  $j$ th material for  $j = 1, 2$ . Using arguments based upon elementary probability theory we find<sup>7</sup>

$$K(r_1, r_2) = \langle \rho_1^2 \rangle \exp \left\{ \frac{-3S |r_1 - r_2|}{8c(1-c)V} \right\} \quad \dots(23)$$

where  $c$  is the volume concentration of the material forming phase 1,  $S$  is its surface area and  $V$  is the total volume. Thus we can assume the following form

$$K(r_1, r_2) = \langle \rho_1^2 \rangle \exp \{-\alpha |r_1 - r_2|\} \quad \dots(24)$$

where  $\alpha$  is a parameter depending upon nature of the composite material.

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## FLOW OF A CONDUCTING FLUID BETWEEN TWO COAXIAL ROTATING POROUS CYLINDERS BOUNDED BY A PERMEABLE BED

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The steady flow of a viscous incompressible conducting fluid between two coaxial rotating porous cylinders with the outer cylinder bounded by a permeable bed is considered. The induced magnetic field is neglected and on the porous bed the boundary condition of Beavers and Joseph is applied. An exact solution of the governing equations is found. The velocity and temperature distribution of the fluid in free region and in porous region are evaluated in dimensionless form. The results are discussed numerically.

### 1. INTRODUCTION

Flow through and past porous media have wider range of applications in many branches like chemical engineering, petroleum engineering, soil mechanics and biomedical engineering.

The study of the flow of a viscous incompressible fluid between two coaxial cylinders was first undertaken by Couette<sup>2</sup> with a view to measuring the viscosity of the fluid. Sinha and Choudhary<sup>6</sup> and Jain and Bansal<sup>3</sup> consider the flow of a viscous incompressible fluid between two coaxial rotating or non rotating cylinders with the walls of the cylinder being either solid or porous. Jain and Bansal<sup>3</sup> considered the flow of a viscous incompressible fluid between two coaxial rotating porous cylinders. Jain and Mehta<sup>4</sup> obtained exact solution in a closed form of the hydro magnetic equations for an incompressible viscous and electrically conducting fluid flow through an annulus with porous walls in the presence of a transverse radial magnetic field. Syam Babu<sup>7</sup> discussed the flow of a viscous incompressible conducting fluid between two coaxial rotating porous cylinders under the influence of a uniform radial magnetic field.

In this paper we study the flow of viscous conducting incompressible fluid between two coaxial rotating porous cylinders with the outer cylinder bounded by a permeable bed. The cylinders are rotating with angular velocities  $\omega_1$  and  $\omega_2$  res-

pectively in the same direction. The fluid is injected/sucked at the inner cylinder with a constant velocity. The flow in the annulus and in the porous medium is governed by the same pressure gradient  $-dp/dr$ .

## 2. NOMENCLATURE

$(u, v, 0)$	= dimensionless fluid velocity in the free region
$(u_1, v_1, 0)$	= dimensionless fluid velocity in the porous region
$(Hr, 0, 0)$	= magnetic field strength
$(0, 0, Ez)$	= electric field strength
$\rho$	= density of the fluid
$\nu$	= kinematic viscosity of the fluid
$C_p$	= specific heat at constant pressure
$\kappa$	= thermal conductivity of the fluid in zone 1
$T$	= temperature in the fluid region
$T_0$	= ambient temperature
$h_e$	= heat transfer coefficient
$\mu$	= coefficient of viscosity
$P$	= fluid pressure
$\sigma$	= porosity parameter
$K$	= permeability of the porous bed
$L$	= dimensionless constant
$\Omega_1, \Omega_2$	= angular velocities of the inner and outer cylinders respectively
$Pr$	= prandtl number
$Pe$	= peclet number
$Nu$	= nusselt number
$\mu_e$	= magnetic permeability of the fluid
$\sigma_e$	= electrical conductivity of the fluid
$\delta$	= dimensionless parameter
$M$	= magnetic parameter
$\phi$	= dissipation function



- $r_1, r_2$  = radii of the inner and outer cylinders respectively  
 $\lambda$  = suction/injection parameter  
 $\alpha$  = slip parameter.

### 3. FORMULATION OF THE PROBLEM

We consider the steady flow of a conducting viscous incompressible fluid between two coaxial rotating porous cylinders composed of an insulated material. The cylinders terminate at perfect electrodes which are connected through a load. The walls of the cylinders being porous with the outer cylinder bounded by a permeable bed. (Fig. 1) The problem is divided into two zones : free zone and porous zone. In zone 1, the fluid is governed by the magneto hydrodynamic equations and in zone 2, the flow is governed by the Darcy's law. The solutions in two zones are matched at the interface

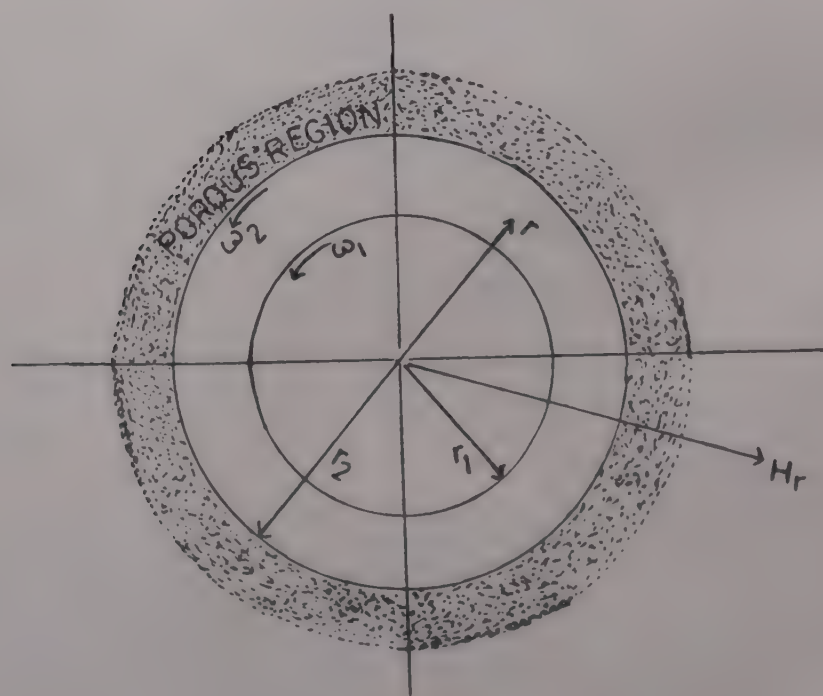


FIG. 1. Physical model.

between zone 1 and zone 2 by assuming the slip velocity boundary condition of Beavers and Joseph<sup>1</sup>. A uniform magnetic field  $H_0$  is applied in the radial direction throughout the flow region. The porous zone is assumed as isothermal and so the bed is kept at constant temperature  $T_0$  which is ambient temperature. The boundary condition of Rudraiah *et al.*<sup>5</sup>, is used for temperature at the interface between the porous zone and free zone.

The governing equations for the steady viscous, incompressible fluid flow in zone 1 and zone 2 are :

## Zone 1

$$u' \frac{du'}{dr'} - \frac{v'^2}{r'} = - \frac{1}{\rho} \frac{dp'}{dr'} + \nu \left( \frac{d^2 u'}{dr'^2} + \frac{1}{r'} \frac{du'}{dr'} - \frac{u'}{r'^2} \right) \quad \dots(1)$$

$$u' \frac{v'}{dr'} + \frac{u' v'}{r'} = \nu \left( \frac{d^2 v'}{dr'^2} + \frac{1}{r'} \frac{dv'}{dr'} - \frac{v'}{r'^2} \right) + \frac{1}{\rho} (\sigma_e \mu_e H_0 Ez' - \sigma_e \mu_e^2 H_0^2 v') \quad \dots(2)$$

$$\frac{d(r' u')}{dr'} = 0 \quad \dots(3)$$

$$\frac{d(r' Hr')}{dr'} = 0 \quad \dots(4)$$

$$\frac{d(Ez')}{dr'} = 0 \quad \dots(5)$$

$$C_p u' \frac{dT'}{dr'} = \frac{\kappa}{\rho} \left( \frac{d^2 T'}{dr'^2} + \frac{1}{r'} \frac{dT'}{dr'} \right) + \phi \quad \dots(6)$$

where

$$\phi = \nu \left[ 2 \left\{ \left( \frac{du'}{dr'} \right)^2 + \left( \frac{u'}{r'} \right)^2 \right\} + \left( \frac{dv'}{dr'} - \frac{v'}{r'} \right)^2 \right]$$

## Zone 2

$$u'_1 = - \frac{K}{\mu} \left[ \frac{dp'}{dr'} - 2\rho \omega_2 v'_1 \right] \quad \dots(7)$$

$$v'_1 = - \frac{K}{\mu} \left[ 2\rho \omega_2 u'_1 - \sigma_e \mu_e H_0 Ez' + \sigma_e \mu_e^2 H_0^2 v'_1 \right] \quad \dots(8)$$

$$\frac{d(r' u'_1)}{dr'} = 0. \quad \dots(9)$$

The boundary conditions are

$$u' = u_1, v' = r_1 \omega_1, T' = T_1 \text{ at } r' = r_1 \quad \dots(10)$$

$$\left. \begin{aligned} u' &= u'_{b1}, \quad \frac{du'}{dr'} = \frac{\alpha}{\sqrt{K}} (u'_{b1} - u'_1) \\ v' &= v'_{b1} + r_2 \omega_2, \quad \frac{dv'}{dr'} = \frac{\alpha}{\sqrt{K}} (v'_{b1} - v'_1) \\ \frac{dT'}{dr'} &= \frac{h_e}{\kappa} (T'_B - T_0) \end{aligned} \right\} \text{ at } r' = r_2 \quad \dots(11)$$

The following non-dimensional quantities are used :

$$r = \frac{r'}{r_1}, u = \frac{u' r_1}{v}, v = \frac{v' r_1}{v}, u_1 = \frac{u'_1 r_2}{v}, v_1 = \frac{v'_1 r_2}{v},$$

$$u_{b1} = \frac{u'_{b1} r_2}{v}, v_{b1} = \frac{v'_{b1} r_2}{v}, P = \frac{P' r_1^2}{\rho v^2}, \delta = \frac{r_2}{r_1}, \Omega_1 = \frac{r_1^2 \omega_1}{v},$$

$$\Omega_2 = \frac{r_2^2 \omega_2}{v}, T_B = \frac{T'_B - T_0}{T_2 - T_0}, \lambda = \frac{r_1 u}{v} = \frac{r_2 u_1}{v}, Pr = \frac{\mu C_p}{\kappa},$$

$$Pe = \lambda Pr, L = \frac{\mu v^2}{r_1^2 \kappa (T_2 - T_0)}, Ez = \frac{Ez'}{E_0}, \text{ where } E_0 = \frac{\mu_e H_0 v}{r_1}$$

$$Hr = \frac{Hr'}{H_0}, \sigma = \frac{r_2}{\sqrt{K}}, Nu = \frac{h_e r_1}{\kappa}, M^2 = \frac{\sigma_e \mu_e^2 H_0^2 r_1^2}{\mu}.$$

Using the above non-dimensional quantities, eqns. (1) – (9) reduce to :

Zone 1

$$u \frac{du}{dr} - \frac{v^2}{r} = - \frac{dp}{dr} + \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} \quad \dots(12)$$

$$u \frac{dv}{dr} + \frac{uv}{r} = \frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \left( \frac{1}{r^2} + \frac{M^2 Hr^2}{r^2} \right) v + \frac{M^2 Hr Ez}{r} \quad \dots(13)$$

$$\frac{d(ru)}{dr} = 0, \frac{d(rHr)}{dr} = 0, \frac{d(Ez)}{dr} = 0 \quad \dots(14)$$

$$Pr u \frac{dT}{dr} = \frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} + L \left[ 2 \left\{ \left( \frac{du}{dr} \right)^2 + \left( \frac{u}{r} \right)^2 \right\} + \left( \frac{dv}{dr} - \frac{v}{r} \right)^2 \right] \quad \dots(15)$$

Zone 2

$$u_1 = - \frac{1}{\sigma^2} \left[ \delta^3 \frac{dp}{dr} - 2 v_1 \Omega_2 \right] \quad \dots(16)$$

$$v_1 = \frac{M^2 \delta^3 Hr Ez - 2 \Omega_2 u_1}{\sigma^2 + \delta^2 M^2 Hr^2} \quad \dots(17)$$

$$\frac{d(ru_1)}{dr} = 0. \quad \dots(18)$$



The boundary conditions (10) – (11) reduce to

$$u = \lambda, v = \Omega_1, T = 0 \quad \text{at } r = 1 \quad \dots (19)$$

$$\left. \begin{aligned} u &= u_{b1}, \frac{du}{dr} = \frac{\alpha \sigma}{\delta^2} (u_{b1} - u_1) \\ v &= v_{b1} + \Omega_2 \delta, \frac{dv}{dr} = -\frac{\alpha \sigma}{\delta^2} (v_{b1} - v_1) \\ T &= T_B, \frac{dT}{dr} = Nu T_B. \end{aligned} \right\} \quad \text{at } r = \delta \quad \dots (20)$$

#### 4. VELOCITY DISTRIBUTION

Solving equations (12) – (14) and (16) – (18) with the help of the boundary conditions (19) and (20), we get

$$u = \lambda/r$$

$$v = C r^{N_1} + D r^{N_2} + C_1 r \quad \dots (22)$$

$$u_1 = \frac{B}{r} \quad \dots (23)$$

$$v_1 = \frac{M^2 r \delta^3 E z - 2 B r \Omega_2}{r^2 \sigma^2 + M^2 \delta^2} \quad \dots (24)$$

where

$$B = \lambda [1 + \delta/\alpha\sigma], C_1 = \frac{M^2 E z}{M^2 + 2\lambda}$$

$$N_{1,2} = \frac{1}{2} [\lambda \pm g \{\lambda^2 + 4(1 + M^2 + \lambda)^{1/2}\}].$$

The constants  $C$  and  $D$  are functions of the physical parameters involving in the problem.

#### 5. TEMPERATURE DISTRIBUTION

Using the velocity of the fluid obtained in zone 1, equation (15) reduces to

$$r^2 \frac{d^2 T}{dr^2} + (1 - Pe) r \frac{dT}{dr} = -\frac{C_0}{r^2} - C_2 r^{2N_1} - C_3 r^{2N_2} - C_4 r^\lambda$$

where

$$C_0 = 4 L \lambda^2, C_2 = LC^2 (N_1 - 1)^2, C_3 = LD^2 (N_2 - 1)^2$$

$$C_4 = 2 LCD (N_1 - 1) (N_2 - 1).$$

This is a second order ordinary differential equation whose general solution will possess a number of singular points. In order to avoid these singularities the solutions are obtained for separate cases by using the boundary conditions (19) – (20).

(i) *No singularities* :  $[Pe \neq -2, 2N_1, 2N_2, \lambda, \lambda \neq -(1 + M^2)]$

$$T = \frac{C_0 (1 - r^{-2})}{2 (2 + Pe)} + \frac{C_2 (1 - r^{2N_1})}{2N_1 (2N_1 - Pe)} + \frac{C_3 (1 - r^{2N_2})}{2N_2 (2N_2 - Pe)} \\ + \frac{C_4 (1 - r^\lambda)}{\lambda (\lambda - Pe)} + B_2 (1 - r^{Pe}); \quad \dots(26)$$

(ii) *Singularities* :  $[Pe = -2, Pe \neq 2N_1, 2N_2, \lambda, \lambda \neq -(1 + M^2)]$

$$T = \frac{C_0 r^{-2} \log r}{2} + \frac{C_2 (1 - r^{2N_1})}{2N_1 (2N_1 + 2)} + \frac{C_3 (1 - r^{2N_2})}{2N_2 (2N_2 + 2)} \\ + \frac{C_4 (1 - r^\lambda)}{\lambda (\lambda + 2)} + B_3 (1 - r^{-2}); \quad \dots(27)$$

(iii)  $[Pe = -2 = \lambda, Pe \neq 2N_1, 2N_2, M \neq 1]$

$$T = \frac{C_2 (1 - r^{2N_1})}{2N_1 (2N_1 + 2)} + \frac{C_3 (1 - r^{2N_2})}{2N_2 (2N_2 + 2)} + \frac{C' r^{-2} \log r}{2} + B_4 (r^{-2} - 1); \quad \dots(28)$$

(iv)  $[Pe = -2 = \lambda, Pe \neq 2N_2, 2N_1 = 0, M = 1, Pe \neq 2N_1]$

$$T = \frac{C' r^{-2} \log r}{2} - \frac{C_2 \log r}{2} + \frac{C_3 (1 - r^{2N_2})}{2N_2 (2N_2 + 2)} + B_5 (r^{-2} - 1); \quad \dots(29)$$

(v)  $[Pe \neq \lambda, \lambda = -(1 + M^2), 2N_1 = 0, Pe \neq 2N_2, 2N_1]$

$$T = \frac{C_0 (1 - r^{-2})}{2 (2 + Pe)} + \frac{C_2 \log r}{Pe} + \frac{C_3 (1 - r^{2N_2})}{2N_2 (2N_2 - Pe)} \\ + \frac{C_4 (1 - r^\lambda)}{\lambda (\lambda - Pe)} + B_6 (r^{Pe} - 1); \quad \dots(30)$$

(vi)  $[Pe = \lambda = -(1 + M^2), Pe \neq 2N_1, 2N_2; 2N_1 = 0, M \neq 1 \text{ or } \lambda \neq -2]$

$$T = \frac{C_0 (1 - r^{-2})}{2 (1 - M^2)} - \frac{C_2 \log r}{1 + M^2} + \frac{C_3 (1 - r^{2N_2})}{2N_2 (2N_2 + 1 + M^2)} \\ + \frac{C_4 \log r r^{-(1+M^2)}}{1 + M^2} + B_7 (r^{-(1+M^2)} - 1); \quad \dots(31)$$

(vii)  $[Pe \neq -2, 2N_2, \lambda; Pe = 2N_1, 2N_1, 2N_1 \neq 0]$

$$T = \frac{C_0 (1 - r^{-2})}{2 (2 + Pe)} - \frac{C_2 r^{Pe} \log r}{Pe} + \frac{C_3 (1 - r^{2N_2})}{2N_2 (2N_2 - Pe)}$$

(equation continued on p. 533)

$$+ \frac{C_4 (1 - r^\lambda)}{\lambda (\lambda - Pe)} + B_8 (r^{Pe} - 1); \quad \dots(32)$$

where  $B_2 \dots B_8$  are known constants which are functions of the physical parameters involving in the problem.

## 6. DISCUSSION

The velocity distribution obtained from eqn. (22) and temperature distribution obtained from eqns. (26) – (32) in the presence of porous media are evaluated numerically and the results are shown graphically. Figures 2 – 5 gives the velocity

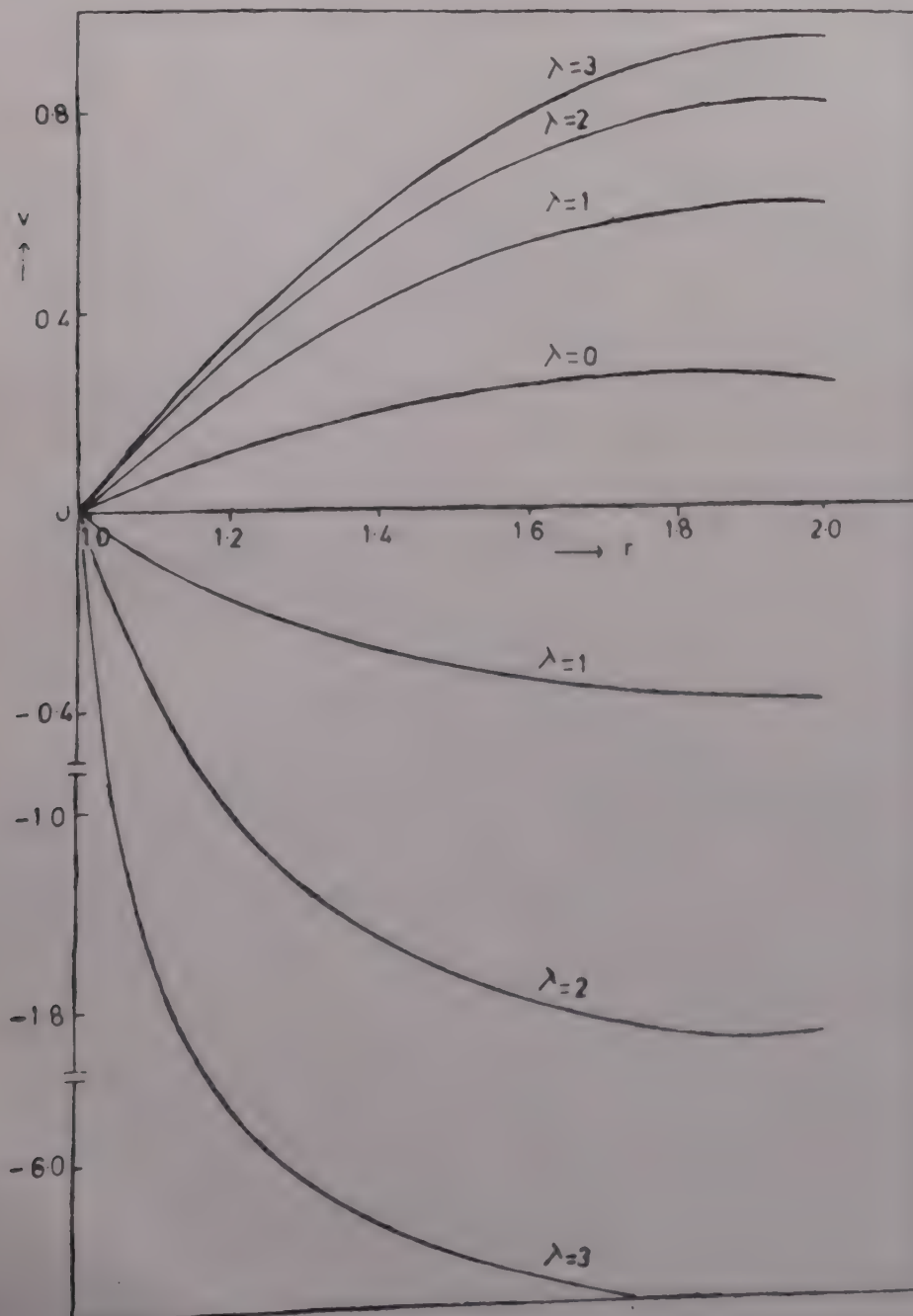


FIG. 2. Velocity profiles for  $C_1 = 1$ ,  $\Omega_1 = \Omega_2 = 0$ ,  $\delta = 2$ ,  $M = 1$  and  $\sigma = 0.05$ .



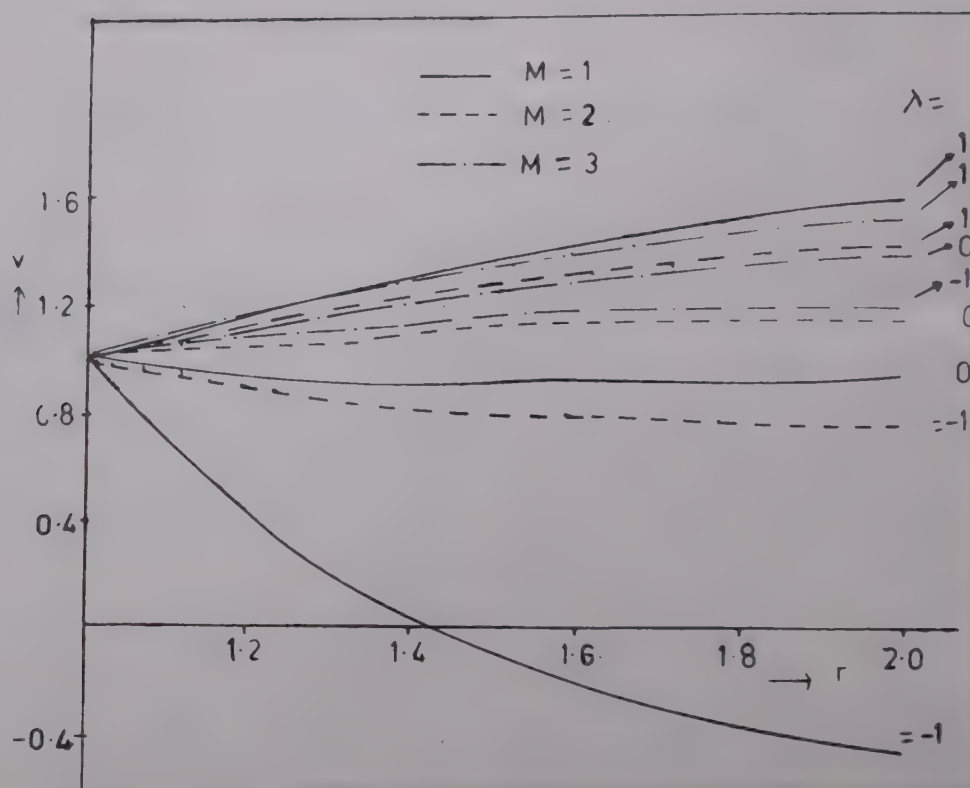


FIG. 3. Velocity profiles for  $C_1 = 1$ ,  $\sigma = 0.05$ ,  $\Omega_1 = \Omega_2 = 1$ ,  $\delta = 2$ .

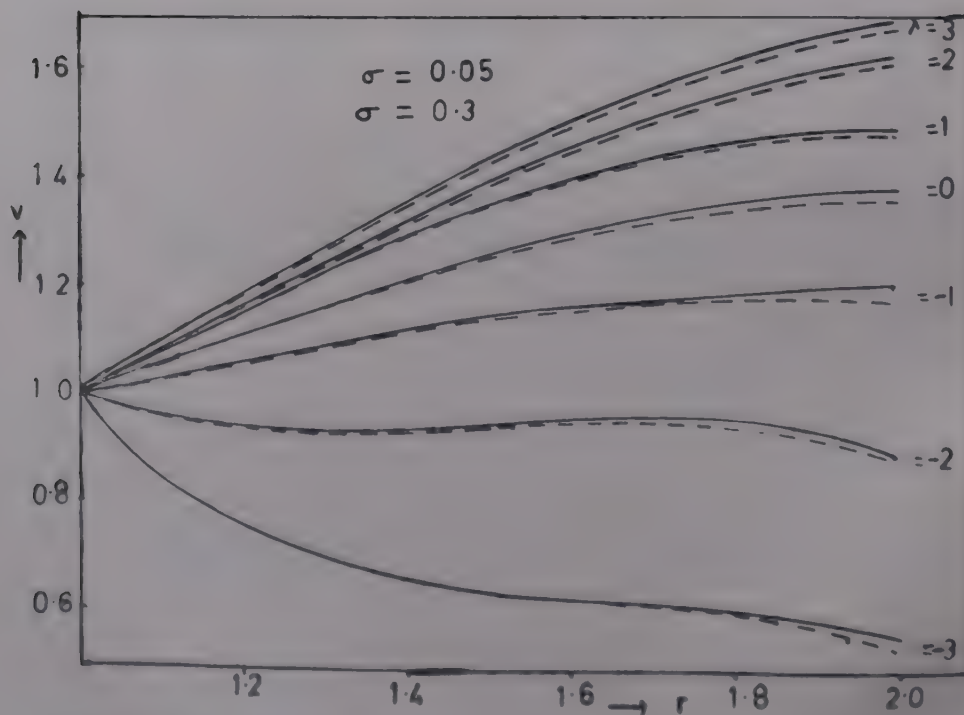


FIG. 4. Velocity profiles for  $C_1 = 1$ ,  $\Omega_1 = \Omega_2 = 1$ ,  $M = 3$ ,  $\delta = 2$ .

distribution for various parameters involving in the problem and Fig. 6 gives the temperature distribution for fixed values of Peclet number and Nusselt number.

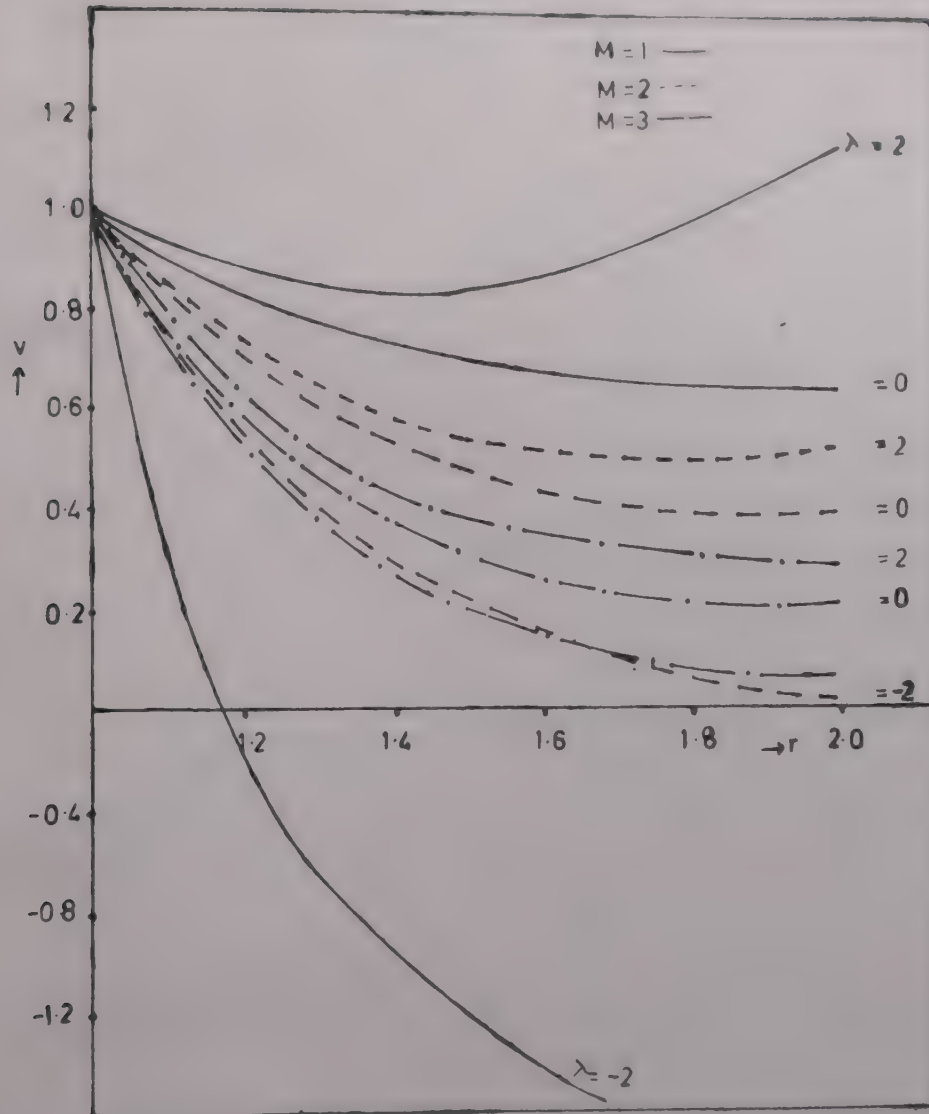


FIG. 5. Velocity profiles for  $\sigma = 0.05$ ,  $\Omega_1 = \Omega_2 = 1$ ,  $C_1 = 0$ .

Figures 2-4 give the velocity profiles in the presence of an electric field for various values of suction/injection parameter  $\lambda$ . When both the cylinders are stationary or rotating the velocity increases as the suction/injection parameter increases. The velocity is negative for suction and positive for injection. When the porosity parameter ' $\sigma$ ' increases the velocity increases for suction and decreases for injection for the fixed values of the magnetic parameter. When both the cylinders are rotating, the velocity increases as magnetic parameter  $M$  increases and for  $\lambda = -1$  the velocity is negative. For small values of  $M$  the velocity is negative ( $M = 1$ ) and for large values of  $M$  the velocity is positive. To remove the back flow, the magnetic parameter is to be increased. For  $M = 3$ , the velocity decreases as porosity parameter increases.

Figure 5 gives the velocity profiles in the absence of electric field. When both the cylinders are rotating the velocity decreases with increasing  $M$  and increases as  $\lambda$  increases. For  $M = 1$ , and  $\lambda = -2$  the velocity is negative and for all other values it is positive.

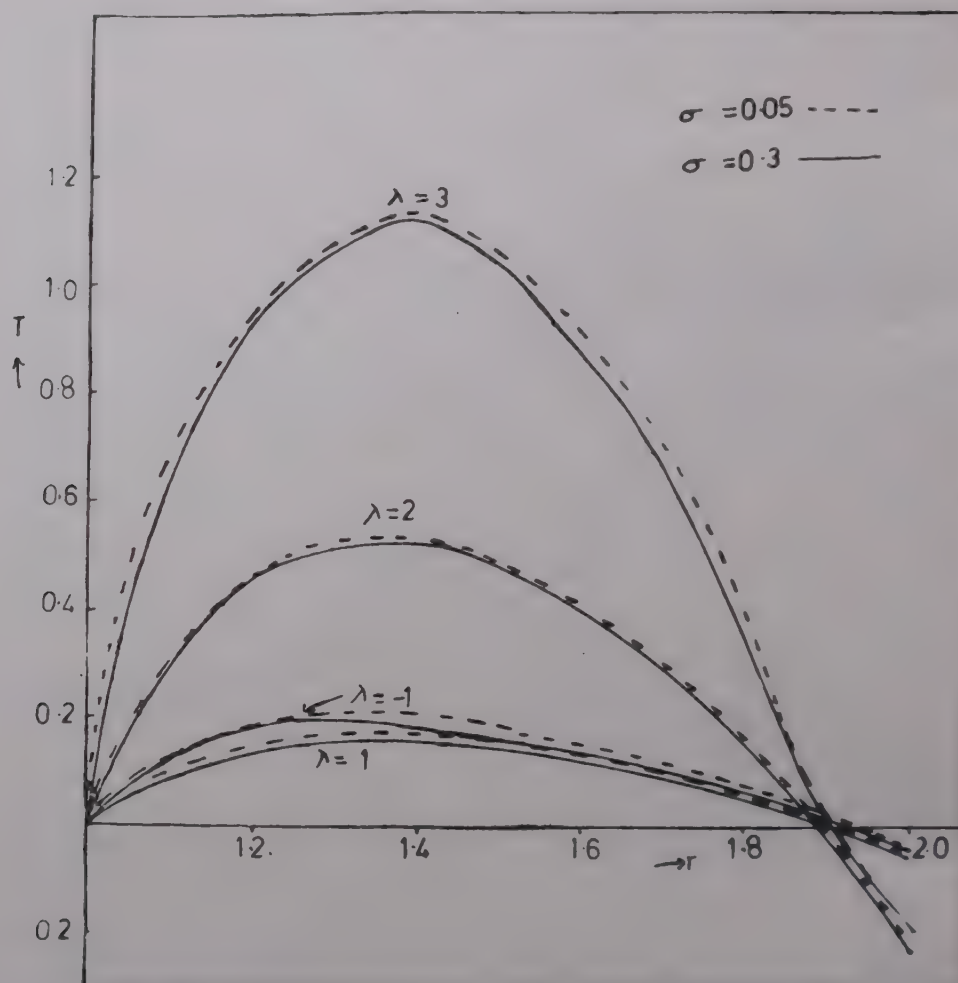


FIG. 6. Temperature profiles for  $C_1 = 1$ ,  $M = 1$ ,  $\Omega_1 = 0$ ,  $\Omega_2 = 1$ .

Figure 6 gives the temperature distribution against ' $r$ ' in the presence of electric field. When the inner cylinder is stationary and the outer cylinder is rotating the temperature is maximum for  $\lambda = -3$  and minimum for  $\lambda = -2$  and  $\lambda = 0$ . The temperature increases with the injection parameter. When the porosity parameter increases the temperature decreases.

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